Central elements in $\mathrm{U}(gl(n))$, shifted symmetric functions, and the superalgebraic Capelli's method of virtual variables

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Abstract

We propose a new method for a unified study of some of the main features of the theory of the center $\boldsymbol{\zeta}(n)$ of the enveloping algebra U(gl(n)) and of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials, that allows the whole theory to be developed, in a transparent and concise way, from the representation-theoretic point of view, that is entirely in the center of U(gl(n)). Our methodological innovation is the systematic use of the superalgebraic method of virtual variables for gl(n), which is, in turn, an extension of Capelli's method of "variabili ausiliarie".

The passage $n \to \infty$ for the algebras $\boldsymbol{\zeta}(n)$ and $\Lambda^*(n)$ is here obtained both as direct and inverse limit in the category of filtered algebras.

The present approach leads to proofs that are almost direct consequences of the definitions and constructions: they often reduce to a few lines computation.

Keyword: Combinatorial representation theory; shifted symmetric functions; superalgebras; central elements in U(gl(n)); Capelli identities; superstandard Young tableaux; Schur supermodules.

Contents

1 Introduction

The study of the center $\boldsymbol{\zeta}(n)$ of the enveloping algebra $\mathbf{U}(gl(n))$ of the general linear Lie algebra $gl(n, \mathbb{C})$, and the study of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials have noble and rather independent origins and motivations.

The theme of central elements in $\mathbf{U}(gl(n))$ is a standard one in the general theory of Lie algebras, see e.g. [30]. It is an old and actual one, since it is an offspring of the celebrated Capelli identity ([21], [25], [39], [40], [60], [68], [71]), relates to its modern generalizations and applications ([4], [44], [45], [51], [52], [54], [55], [64], [67]) as well as to the theory of *Yangians* (see, e.g. [49], [50], [53]).

The algebra $\Lambda^*(n)$ of shifted symmetric polynomials is a remarkable deformation of the algebra $\Lambda(n)$ of symmetric polynomials and its study fits into the mainstream of generalizations of the classical theory (e.g., *factorial symmetric functions*, [8], [9], [26], [36], [37], [47], [48]).

Since the algebras $\boldsymbol{\zeta}(n)$ and $\Lambda^*(n)$ are related by the Harish-Chandra isomorphism χ_n (see, e.g. [56]), their investigation can be essentially regarded as a single topic, and this fact gave rise to a fruitful interplay between representation-theoretic methods (e.g., eigenvalues on irreducible representations) and combinatorial techniques (e.g., generating functions).

In this work, we propose a new method for a unified study of some of the main features of the theory of the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$ and of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials that allows the whole theory to be developed, in a transparent and concise way, from the representation-theoretic point of view, that is entirely in the center $\boldsymbol{\zeta}(n)$.

Our methodological innovation is the systematic use of the *superalgebraic method of virtual variables* for $gl(n)$ (see, e.g. [12]), which is, in turn, an extension of Capelli's method of *variabili ausilarie*. Capelli introduced the method of *variabili ausilarie* in order to manage symmetrizer operators in terms of polarization operators, to simplify the study of some skew-symmetrizer operators (namely, the famous central Capelli operator) and developed this idea in a systematic way in his beautiful treatise [25]. Capelli's idea was well suited to treat symmetrization, but it did not work in the same efficient way while dealing with skew-symmetrization.

One had to wait the introduction of the notion of *superalgebras* (see,e.g. [65], [35], [41], [70]) to have the right conceptual framework to treat symmetry and skew-symmetry in one and the same way. To the best of our knowledge, the first mathematician who intuited the connection between Capelli's idea and superalgebras was Koszul in 1981 [46]; Koszul proved that the classical determinantal Capelli operator can be rewritten - in a much simpler way - by adding to the symbols to be dealt with an extra auxiliary symbol that obeys to different commutation relations.

The *superalgebraic/supersymmetric method of virtual variables* was developed in its full extent and generality (for the general linear Lie superalgebras $gl(m|n)$ - in the notation of [41]) in the series of notes [14], [15], [16], [17], [13] by the present authors and R.Q. Huang.

The superalgebraic method of virtual variables allows remarkable classes of elements in $\boldsymbol{\zeta}(n)$ to be written as simple *sums* of *monomials* and their actions on irreducible representations to be given simple combinatorial descriptions; moreover, this method throws a bridge between the theory of $\boldsymbol{\zeta}(n)$ and the *(super)straightening techniques* [38], [16], [17], [18], [11] (or, in the classical context, *standard monomial theory*, see e.g. [60]).

We consider five classes of central elements, which arise in a natural way in the context of the virtual method when dealing with symmetry and skew-symmetry in $\mathbf{U}(gl(n))$:

- The elements $\mathbf{H}_k(n), \quad k = 1, 2, \ldots, n$; a noteworthy fact is that these elements turn out to be a supple form of the classical determinantal Capelli elements of 1893 [1] ([23], [24], [18]).

- The elements $\mathbf{I}_k(n), \quad k \in \mathbf{Z}^+$; a noteworthy fact is that these elements turn out to be a supple form of the central elements studied by Nazarov [52] and Umeda [69].

- The elements $\mathbf{K}_\lambda(n)$ and $\mathbf{J}_\lambda(n)$, that are generalizations to Young shapes λ of the $\mathbf{H}_k(n)$ and of the $\mathbf{I}_k(n)$, respectively. The actions of these elements on irreducible representations have remarkable triangularity properties (Theorem 4.32 and Proposition 4.39).

[1] These elements are different from the Capelli elements in the sense, for example, of Howe and Umeda [40], even if the two families are closely related (see, e.g. [12]).

- The elements $\mathbf{S}_\lambda(n)$, that provide a common generalization of both $\mathbf{H}_k(n)$ and $\mathbf{I}_k(n)$. By Theorem 4.51 and the Sahi/Okounkov Characterization Theorem (see Subsection 4.5.3 below), the elements $\mathbf{S}_\lambda(n)$ turn out to be a supple form of the *Schur elements* discovered by Sahi [62] in the context of shifted symmetric polynomials, by Okounkov [54] , [55] as elements of $\boldsymbol{\zeta}(n)$, and extensively investigated by Okounkov and Olshanski [56] both from the point of view of central elements and of shifted symmetric functions in infinitely many variables.

Just to mention a few remarkable features of the method, we first note that the centrality of the elements we consider follows from extremely simple arguments on their virtual presentations (see, e.g. Proposition 4.3 and Proposition 4.23). The duality/reciprocity in $\boldsymbol{\zeta}(n)$ (Theorem 4.59) immediately follows from a new (and rather surprising) combinatorial description of the eigenvalues of the Capelli elements $\mathbf{H}_k(n)$ on irreducible representations (Proposition 4.7) that is *dual* (in the sense of shapes/partitions) to the combinatorial description of the eigenvalues of the Nazarov/Umeda elements $\mathbf{I}_k(n)$ (Theorem 4.25.1). By the *Bitableaux correspondence and Koszul map Theorems* ([17], Thms. 1 and 2, see also [12], [46]), the elements $\mathbf{H}_k(n), \mathbf{I}_k(n), \mathbf{K}_\lambda(n), \mathbf{J}_\lambda(n)$ expand into "*column bitableaux*" in $\mathbf{U}(gl(n))$ in a way that is in all respect similar to the ordinary Laplace expansions of determinants and permanents of matrices with entries in a commutative algebra. This fact, in turn, leads to further combinatorial descriptions of the eigenvalues of the central elements $\mathbf{H}_k(n), \mathbf{I}_k(n), \mathbf{K}_\lambda(n), \mathbf{J}_\lambda(n)$ on irreducible representations that make apparent the role of permutations (for the sake of brevity, we fully work out only the case of the $\mathbf{H}_k(n)$'s, in Subsection 4.1.2). Our representation-theoretic versions (Theorem 4.50 and Theorem 4.51) of the Okounkov Vanishing Theorem and of the Sahi/Okounkov Characterization Theorem follow at once from some standard elementary facts on Schur supermodules (Proposition 3.37) in combination with Regonati's hook lemma [61] (Proposition 3.38).

The passage to the infinite dimensional case $n \to \infty$ for the algebras $\boldsymbol{\zeta}(n)$ is rather subtle; the "naive" $\infty-$dimensional analogue of the algebras $\mathbf{U}(gl(n))$, that is the direct limit algebra $\varinjlim \mathbf{U}(gl(n))$ with respect to the *"inclusion" monomorphisms*, has trivial center.

The $\infty-$dimensional analogue $\boldsymbol{\zeta}$ of the algebras $\boldsymbol{\zeta}(n)$ is here obtained as the direct limit algebra $\varinjlim \boldsymbol{\zeta}(n)$ (in the category of filtered algebras) with respect to a family of monomorphisms $\mathbf{i}_{n+1,n} : \boldsymbol{\zeta}(n) \hookrightarrow \boldsymbol{\zeta}(n+1)$, that we call the *Capelli monomorphisms* (Section 5.1). This construction is a Lie algebra analogue of the construction of the ring of symmetric functions Λ as the direct limit algebra $\varinjlim \Lambda(n)$ with respect to the monomorphims that map the elementary symmetric polynomial $e_k(n)$ in n variables to the elementary symmetric polynomial $e_k(m)$ in m variables, $n < m$, $k = 1, 2, \ldots, n$. (see, e.g. [34], or `<https://en.wikipedia.org/wiki/Ring_of_symmetric_functions>`).

The direct limit construction of the algebra $\boldsymbol{\zeta}$ implies that, if $P = \varinjlim P(n)$ and $Q = \varinjlim Q(n)$ are elements of $\boldsymbol{\zeta}$ of "minimum filtration degree" m, then $P = Q$ if and only if $P(n) = Q(n)$ in $\boldsymbol{\zeta}(n)$, for some $n \geq m$; therefore, linear and algebraic relations among elements of $\boldsymbol{\zeta}$ are determined by the relations among their "*germs*" in $\boldsymbol{\zeta}(n)$, for n sufficiently large.

An intrinsic/invariant presentation of the Capelli monomorphisms is obtained, in Section 5.2, via a family of projections $\boldsymbol{\mu}_{n,n+1} : \boldsymbol{\zeta}(n+1) \twoheadrightarrow \boldsymbol{\zeta}(n)$, the *Olshanski projections* [57], [59] (see also [49]). The Olshanski projections $\boldsymbol{\mu}_{n,n+1}$ are *left* inverses of the Capelli monomorphisms $\mathbf{i}_{n+1,n}$, and they become *two-sided* inverses when restricted to the filtration elements $\boldsymbol{\zeta}(n+1)^{(m)}$ and $\boldsymbol{\zeta}(n)^{(m)}$, for n sufficiently large (Proposition 5.4 and Proposition 5.8).

Amazingly, the Olshanski projection $\boldsymbol{\mu}_{n,n+1}$ acts on virtual presentations just by truncating the sums of monomials (see Proposition 5.6 and Proposition 5.10). Therefore, the direct limits $\mathbf{H}_k$ of the $\mathbf{H}_k(n)$, $\mathbf{I}_k$ of the $\mathbf{I}_k(n)$, $\mathbf{K}_\lambda$ of the $\mathbf{K}_\lambda(n)$, $\mathbf{J}_\lambda$ of the $\mathbf{J}_\lambda(n)$, and $\mathbf{S}_\lambda$ of the $\mathbf{S}_\lambda(n)$ can be consistently written as *formal series* of virtual *monomials* (Theorem 5.11, Definition 5.12 and Proposition 5.13).

The interplay between Capelli monomorphisms and Olshanski projections shows the algebra $\boldsymbol{\zeta}$ admits a double presentation, both as a direct limit and as an inverse limit. Being the algebra $\boldsymbol{\zeta}$ isomorphic to the algebra Λ^* of *shifted symmetric functions* (Theorem 7.6), the Olshanski projections are the natural counterpart, in the context of the centers $\boldsymbol{\zeta}(n)$, of the Okounkov-Olshanski *stability principle* for the algebras $\Lambda^*(n)$ of shifted symmetric polynomials [56], the isomorphism $\chi : \boldsymbol{\zeta} \to \Lambda^*$ is indeed the "limit" of the Harish-Chandra isomorphisms χ_n and it admits a transparent representation-theoretic interpretation (Proposition 7.8).

We tried to make the exposition self-contained.
The arguments are based on very few prerequisites: the fact that the classical Capelli elements of 1893 (Section 4.1) provide a free system of algebra generators of $\boldsymbol{\zeta}(n)$ and a small bunch of combinatorial lemmas (Proposition 3.37 and Proposition 3.38). The use of the virtual variables turns all the proofs into almost direct consequences of the definitions and constructions.

2 Synopsis

The paper is organized as follows.

In Chapter 3, we summarize the leading ideas of the present approach, that is the main facts and constructions of the *superalgebraic virtual variables method for $gl(n)$* [12].

As already mentioned, we develop the whole theory from the representation-theoretic point of view, that is our main concern are the eigenvalues of central elements of $\mathbf{U}(gl(n))$ on $gl(n)-$irreducible representations. In order to do this, we embed any "covariant" $gl(n)-$irreducible representation (Schur module $Schur_\lambda(n)$) into an irreducible supermodule $Schur_\lambda(m_0|m_1+n)$ on a suitable general linear superalgebra $gl(m_0|m_1+n)$. Besides the algebras $\mathbf{U}(gl(n)) \subset \mathbf{U}(gl(m_0|m_1+n))$, in Subsection 3.5.1 we single out a third algebra, the *virtual* *algebra* $Virt(m_0|m_1+n)$,

$$\mathbf{U}(gl(n)) \subset Virt(m_0|m_1+n) \subset \mathbf{U}(gl(m_0|m_1+n)),$$

that has the remarkable properties:

- there is a "canonical" epimorphism $\mathfrak{p} : Virt(m_0|m_1+n) \twoheadrightarrow \mathbf{U}(gl(n))$, that we call the *Capelli epimorphism*;

- the Schur $\mathbf{U}(gl(n))-$irreducible module $Schur_\lambda(n)$ is an invariant subspace of the $\mathbf{U}(gl(m_0|m_1+n))-$irreducible supermodule $Schur_\lambda(m_0|m_1+n)$, with respect to the action of the subalgebra $Virt(m_0|m_1+n) \subset \mathbf{U}(gl(m_0|m_1+n))$;

- the action of any element of $Virt(m_0|m_1+n)$ on $Schur_\lambda(n)$ is the same of the action of its image in $\mathbf{U}(gl(n))$ with respect to the Capelli epimorphism (Theorem 3.11).

Therefore, instead of studying the action of an element in $\mathbf{U}(gl(n))$ one can study the action of a preimage of it in $Virt(m_0|m_1+n)$ (called *virtual presentation*). The advantage of virtual presentations is that they are frequently of monomial form, admit quite transparent interpretations and are much easier to be dealt with (see, e.g. [14], [15], [19], [11], [12]), so we even take them as a definition of an element in $\mathbf{U}(gl(n))$.

In order to make the virtual variables method effective, we exhibit a class of nontrivial elements that belong to $Virt(m_0+m_1, n)$, that is *balanced monomials* (Subsection 3.5.3). A quite relevant subclass of balanced monomials arises in connection with pairs of Young tableaux (Section 3.6).

When specialized to the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$, this method reveals further features and benefits:

- the subalgebra $Virt(m_0|m_1+n)$ is an invariant subspace of $\mathbf{U}(gl(m_0|m_1+n))$ with respect to the adjoint action of $gl(n)$;

- the Capelli epimorphism is an equivariant map with respect to the adjoint action of $gl(n)$ (Proposition 3.8);

- the Capelli epimorphism image of an element of $Virt(m_0|m_1+n)$ that is an *invariant* with respect to the adjoint action of $gl(n)$ belongs to the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$ (Corollary 3.9).

Therefore, in Chapter 4 we will systematically define classes of central elements through their virtual presentations; in this way, the centrality is immediately apparent from the definition.

In Section 3.7, we recall a family of algebra generators of the supersymmetric algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$, called *biproducts*, which are supersymmetric generalizations of "formal determinants", as well as a a family of linear generators associated to a pairs of Young tableaux, called *bitableaux*, for short. Bitableaux are signed products of biproducts [38] (for a virtual presentation of both, see [14],

[11]), both share a good behaviour with respect to the superpolarization action of $\mathbf{U}(gl(m_0|m_1+n))$ on $\mathbb{C}[M_{m_0|m_1+n,d}]$ (Proposition 3.27 and Proposition 3.28). By the superstraightening algorithm, the set of (super)standard bitableaux is a basis of $\mathbb{C}[M_{m_0|m_1+n,d}]$ [38], [18], [11] (Theorem 3.30 and Corollary 3.32).

In Section 3.8, we review the main facts about Schur supermodules as submodules of $\mathbb{C}[M_{m_0|m_1+n,d}]$, and of Schur modules as $\mathbf{U}(gl(n))-$submodules of Schur supermodules. The highest weight vectors of Schur modules turns out to be *Deruyts bitableaux* (Subsection 3.8.2).

In Subsection 3.8.3, we recall some results about the action of the Lie superalgebra $\mathbf{U}(gl(m_0|m_1+n))$ on the the subspace $Schur_\lambda(n)$ (*Vanishing Lemmas*, Proposition 3.37 and *Regonati's Hook Lemma*, Proposition 3.38), which are direct consequences of the use of virtual variables in combination with the superstraightening algorithm. These results will play a crucial role in Chapter 4; as a matter of fact, the "triangularity/orthogonality" results for the action of the central elements $\mathbf{K}_\lambda(n)$, $\mathbf{J}_\lambda(n)$, $\mathbf{S}_\lambda(n)$ on highest weight vectors almost immediately follow from them.

In Chapter 4, we introduce five classes $\mathbf{H}_k(n)$, $\mathbf{I}_k(n)$, $\mathbf{K}_\lambda(n)$, $\mathbf{J}_\lambda(n)$, and $\mathbf{S}_\lambda(n)$ of elements of $\mathbf{U}(gl(n))$ as images of *simple sums of balanced monomials* in $Virt(m_0|m_1+n)$. Due to their virtual presentations, these elements are almost immediately recognized as belonging to the center $\boldsymbol{\zeta}(n)$. The $\mathbf{H}_k(n)$'s and the $\mathbf{I}_k(n)$'s turn out to be the Capelli determinantal elements and their permanental analogues, respectively.

In Subsection 4.1.2 we consider the Capelli generators $\mathbf{H}_k(n)$ from the point of view of the *Koszul isomorphism* and show that they expand into "*column bitableaux*" in the same way as the determinants of matrices with commutative entries expand into ordinary monomials (Proposition 4.10); this result implies a new combinatorial description of their eigenvalues on irreducible representations (Proposition 4.15).

The *shaped* generalizations $\mathbf{K}_\lambda(n)$ of $\mathbf{H}_k(n)$ and $\mathbf{J}_\lambda(n)$ of $\mathbf{I}_k(n)$ provide two new linear bases of $\boldsymbol{\zeta}(n)$ (Corollary 4.33 and Corollary 4.42) that satisfy remarkable triangularity properties when acting on highest weight vectors (Theorem 4.32 and Theorem 4.39). From an intuitive point of view, the $\mathbf{K}_\lambda(n)$'s and the $\mathbf{J}_\lambda(n)$'s are elements with "internal" row skew-symmetry and symmetry respectively, as should be clear from their virtual presentations.

In Section 4.5, we introduce a third basis whose elements $\mathbf{S}_\lambda(n)$ have both "internal" row skew-symmetry and column symmetry. Their action on highest weight vectors (Theorem 4.51) satisfy the condition of the Sahi-Okounkov Characterization Theorem and, therefore, the $\mathbf{S}_\lambda(n)$'s turn out to be the *Schur elements* described by Okounkov as *quantum immanants* (for further details, see Subsection 4.5.3). It is remarkable that the same elements can also be obtained by interchanging the symmetries, that is the $\mathbf{S}_\lambda(n)$ can be defined as having column skew-symmetry and row symmetry (Subsection **??**).

In Section 4.6 we deal with *duality* in $\boldsymbol{\zeta}(n)$[2], which is a Lie algebra analogue

[2]Equivalent results - in the sense of the Harish-Chandra isomorphism - were obtained by Okounkov and Olshanski [56] in the context of the algebra $\Lambda^*(n)$ of shifted symmetric

of the classical involution of the algebra $\Lambda(n)$ of symmetric polynomials. The algebra $\boldsymbol{\zeta}(n)$ has an involution $\mathcal{W}_n$ with notable properties:

- If $\lambda_1,\ \widetilde{\lambda}_1 \leq n$, the image $\mathcal{W}_n\big(\mathbf{S}_\lambda(n)\big)$ equals $\mathbf{S}_{\widetilde{\lambda}}(n)$ (Corollary 4.61).
- The eigenvalue of an element $\boldsymbol{\varrho} \in \boldsymbol{\zeta}(n)$ on a highest weight vector of weight μ equals the eigenvalue of its image $\mathcal{W}_n\big(\boldsymbol{\varrho}\big)$ on a highest weight vector of weight $\widetilde{\mu}$ (Theorem 4.59).

In the first Section of Chapter 5, we construct the $\infty-$dimensional analogue $\boldsymbol{\zeta}$ of $\boldsymbol{\zeta}(n)$ as the direct limit (in the category of filtered algebras) with respect to the family of Capelli monomorphism

$$\mathbf{i}_{n+1,n} : \boldsymbol{\zeta}(n) \hookrightarrow \zeta(n+1), \qquad \mathbf{i}_{n+1,n}\big(\mathbf{H}_k(n)\big) = \mathbf{H}_k(n+1), \quad k = 1, 2, \ldots, n.$$

The left inverses of the $\mathbf{i}_{n+1,n}$:

$$\boldsymbol{\mu}_{n,n+1} = \boldsymbol{\pi}_{n,n+1} : \boldsymbol{\zeta}(n+1) \twoheadrightarrow \boldsymbol{\zeta}(n),$$

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{H}_k(n+1)) = \mathbf{H}_k(n),\ k = 1, 2, \ldots, n, \quad \boldsymbol{\pi}_{n,n+1}(\mathbf{H}_{n+1}(n+1)) = 0,$$

become *two-sided* inverses when restricted to the filtration elements $\boldsymbol{\zeta}(n)^{(m)}$ and $\boldsymbol{\zeta}(n+1)^{(m)}$, for n sufficiently large. The main point is that the projections $\boldsymbol{\pi}_{n,n+1}$ admit an intrinsic description in terms of the *Olshanski decomposition* (Section 5.2). Due to their virtual presentations, the Olshanski decompositions of the elements $\mathbf{H}_k(n)$, $\mathbf{I}_k(n)$, $\mathbf{K}_\lambda(n)$, $\mathbf{J}_\lambda(n)$, and $\mathbf{S}_\lambda(n)$ are amazingly simple (Proposition 5.6 and Proposition 5.10) and imply that

$$\mathbf{i}_{n+1,n}\big(\mathbf{I}_k(n)\big) = \mathbf{I}_k(n+1), \qquad \mathbf{i}_{n+1,n}\big(\mathbf{K}_\lambda(n)\big) = \mathbf{K}_\lambda(n+1),$$
$$\mathbf{i}_{n+1,n}\big(\mathbf{J}_\lambda(n)\big) = \mathbf{J}_\lambda(n+1), \qquad \mathbf{i}_{n+1,n}\big(\mathbf{S}_\lambda(n)\big) = \mathbf{S}_\lambda(n+1),$$

for n "sufficiently large" (Section 5.3).

In Chapter 6, we recall the Harish-Chandra isomorphism $\chi_n : \boldsymbol{\zeta}(n) \longrightarrow \Lambda^*(n)$, $\Lambda^*(n)$ the algebra of shifted symmetric polynomials, and translate the results of Chapter 4 from $\boldsymbol{\zeta}(n)$ to $\Lambda^*(n)$.

In Chapter 7, we introduce the isomorphism $\chi : \boldsymbol{\zeta} \longrightarrow \Lambda^*$, Λ^* the algebra of shifted symmetric functions in infinitely many variables, and translate the results of Chapter 5 from $\boldsymbol{\zeta}$ to Λ^*.

3 The method of virtual supersymmetric variables for $\mathbf{U}(gl(n))$

3.1 The Lie algebra $gl(n)$ as a subalgebra of the general linear Lie superalgebra $gl(m_0|m_1+n)$

Given a vector space V_n of dimension n, we will regard it as a subspace of a $\mathbb{Z}_2-$graded vector space $W = W_0 \oplus W_1$, where

$$W_0 = V_{m_0}, \qquad W_1 = V_{m_1} \oplus V_n.$$

polynomials, by using generating functions techniques.

The *auxiliary* vector spaces V_{m_0} and V_{m_1} (informally, we assume that $dim(V_{m_0}) = m_0$ and $dim(V_{m_1}) = m_1$ are "sufficiently large") are called the spaces of *even virtual vectors* and of *odd virtual vectors*, respectively, and V_n is called the space of *odd proper vectors*.

The inclusion $V_n \subset W$ induces a natural embedding of the general linear Lie algebra $gl(n)$ into the general linear Lie *superalgebra* $gl(m_0|m_1+n)$ (see, e.g. [41], [65], [35]).

Let $A_0 = \{\alpha_1, \ldots, \alpha_{m_0}\}$, $A_1 = \{\beta_1, \ldots, \beta_{m_1}\}$, $L = \{x_1, \ldots, x_n\}$ denote distinguished bases of V_{m_0}, V_{m_1} and V_n, respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|\beta_t| = |x_i| = 1 \in \mathbb{Z}_2$.

Let

$$\{e_{a,b}; a, b \in A_0 \cup A_1 \cup L\}, \qquad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$$

be the standard $\mathbb{Z}_2-$homogeneous basis of $gl(m_0|m_1+n)$ provided by the elementary matrices.

The supercommutator of $gl(m_0|m_1+n)$ has the following explicit form:

$$[e_{a,b}, e_{c,d}] = \delta_{bc}\ e_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}\ e_{c,b},$$

$a, b, c, d \in A_0 \cup A_1 \cup L$.

3.2 The commutative algebra $\mathbb{C}[M_{n,d}]$ as a subalgebra of the supersymmetric algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$

The *algebra of algebraic forms in n vector variables of dimension d* is the polynomial algebra in $n \times d$ variables:

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[x_{ij}]_{i=1,\ldots,n;j=1,\ldots,d}$$

where $M_{n,d}$ represents the matrix with n rows and d columns with "generic" entries x_{ij}:

$$M_{n,d} = [x_{ij}]_{i=1,\ldots,n;j=1,\ldots,d} = \begin{bmatrix} x_{11} & \ldots & x_{1d} \\ x_{21} & \ldots & x_{2d} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nd} \end{bmatrix}.$$

In the following, we will write $(x_i|j)$ in place of x_{ij}, and regard the (commutative) algebra $\mathbb{C}[M_{n,d}]$ as a subalgebra of the *supersymmetric algebra*

$$\mathbb{C}[M_{m_0|m_1+n,d}] = \mathbb{C}\big[(\alpha_s|j), (\beta_t|j), (x_i|j)\big]$$

generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j), (\beta_t|j), (x_i|j),\ j = 1, 2, \ldots, d$, where

$$|(\alpha_s|j)| = 1 \in \mathbb{Z}_2 \ \ and \ \ |(\beta_t|j)| = |(x_i|j)| = 0 \in \mathbb{Z}_2,$$

subject to the commutation relations:

$$(a|h)(b|k) = (-1)^{|(a|h)||(b|k)|}\ (b|k)(a|h),$$

for $a, b \in \{\alpha_1, \ldots, \alpha_{m_0}\} \cup \{\beta_1, \ldots, \beta_{m_1}\} \cup \{x_1, x_2, \ldots, x_n\}$.

We have:

$$\begin{aligned}\mathbb{C}[M_{m_0|m_1+n,d}] &\cong \Lambda\left[(\alpha_s|j)\right] \otimes \mathrm{Sym}\left[(\beta_t|j), (x_h|j)\right] \\ &\cong \Lambda\left[W_0 \otimes P_d\right] \otimes \mathrm{Sym}\left[(W_1 \oplus V_n) \otimes P_d\right],\end{aligned}$$

where $P_d = (P_d)_1$ denotes the trivially (odd) $\mathbb{Z}_2-$graded vector space with distinguished basis $\{j;\ j = 1, 2, \ldots, d\}$.

The algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ is a supersymmetric $\mathbb{Z}_2-$graded algebra (superalgebra), whose $\mathbb{Z}_2-$graduation is inherited by the natural one in the exterior algebra.

3.3 Left superderivations and left superpolarizations

A *left superderivation* ($\mathbb{Z}_2-$homogeneous of degree $|D|$) (see, e.g. [35], [65], [41]) on $\mathbb{C}[M_{m_0|m_1+n,d}]$ is an element $D \in End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,d}]]$ ($\mathbb{Z}_2-$homogeneous of degree $|D|$) that satisfies "Leibniz rule"

$$D(P \cdot Q) = D(P) \cdot Q + (-1)^{|D||P|} P \cdot D(Q),$$

for every $\mathbb{Z}_2-$homogeneous of degree $|D|$ element $P \in \mathbb{C}[M_{m_0|m_1+n,d}]$.

Given two symbols $a, b \in A_0 \cup A_1 \cup L$, the *superpolarization* $D_{a,b}$ of b to a is the unique *left* superderivation of $\mathbb{C}[M_{m_0|m_1+n,d}]$ of parity $|D_{a,b}| = |a| + |b| \in \mathbb{Z}_2$ such that

$$D_{a,b}\left((c|j)\right) = \delta_{bc}\ (a|j),\ \ c \in A_0 \cup A_1 \cup L,\ \ j = 1, \ldots, d. \tag{1}$$

Informally, we say that the operator $D_{a,b}$ *annihilates* the symbol b and *creates* the symbol a.

3.4 The superalgebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ as a $\mathbf{U}(gl(m_0|m_1+n))-$module

The map

$$e_{a,b} \to D_{a,b}, \qquad a, b \in A_0 \cup A_1 \cup L.$$

(that send the elementary matrices to the corresponding superpolarizations) is a Lie superalgebra morphism from $gl(m_0|m_1+n)$ to $End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,d}]]$ and, hence, it uniquely defines the morphism (i.e. a representation):

$$\varrho : \mathbf{U}(gl(m_0|m_1+n)) \to End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,d}]].$$

In the following, we always regard the superalgebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ as a $\mathbf{U}(gl(m_0|m_1+n))-$supermodule, with respect to the representation ϱ. We recall that $\mathbb{C}[M_{m_0|m_1+n,d}]$ is a *semisimple* $\mathbf{U}(gl(m_0|m_1+n))-$supermodule, whose irreducible (simple) submodules are - up to isomorphism - *Schur supermodules* ([14], [15], [11], [27]). Clearly, the subalgebra $\mathbb{C}[M_{n,d}]$ is a $gl(n)-$module of $\mathbb{C}[M_{m_0|m_1+n,d}]$.

3.5 The virtual algebra $Virt(m_0 + m_1, n)$ and the virtual presentations of elements in $\mathbf{U}(gl(n))$

3.5.1 The virtual algebra $Virt(m_0+m_1, n)$ as a subalgebra of $\mathbf{U}(gl(m_0|m_1+n))$ and the Capelli devirtualization epimorphism $\mathfrak{p} : Virt(m_0 + m_1, n) \twoheadrightarrow \mathbf{U}(gl(n))$

Definition 3.1. ***(Irregular expressions)*** *We say that a product*

$$e_{a_m b_m} \cdots e_{a_1 b_1} \in \mathbf{U}(gl(m_0|m_1+n))$$

is an irregular expression whenever there exists a right subsequence

$$e_{a_i,b_i} \cdots e_{a_2,b_2} e_{a_1,b_1},$$

$i \leq m$ and a virtual symbol $\gamma \in A_0 \cup A_1$ such that

$$\#\{j; b_j = \gamma, j \leq i\} > \#\{j; a_j = \gamma, j < i\}. \tag{2}$$

The meaning of an irregular expression in terms of the action of $\mathbf{U}(gl(m_0|m_1+n))$ on the algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ is that there exists a virtual symbol γ and a right subsequence in which the symbol γ is annihilated more times than it was already created.

Definition 3.2. ***(The ideal*** $\mathbf{Irr}$***)*** *The left ideal $\mathbf{Irr}$ of $\mathbf{U}(gl(m_0|m_1+n))$ is the left ideal generated by the set of irregular expressions.*

Remark 3.3. *The action of any element of $\mathbf{Irr}$ on the subalgebra $\mathbb{C}[M_{n,d}] \subset \mathbb{C}[M_{m_0|m_1+n,d}]$ - via the representation ϱ - is identically zero.*

Proposition 3.4. *([16], [12]) The sum $\mathbf{U}(gl(0|n)) + \mathbf{Irr}$ is a direct sum of vector subspaces of $\mathbf{U}(gl(m_0|m_1+n))$.*

Definition 3.5. ***(The virtual algebra*** $Virt(m_0+m_1,n)$***)*** *The vector space $Virt(m_0+m_1, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr}$ is a subalgebra of $\mathbf{U}(gl(m_0|m_1+n))$, called the virtual subalgebra [16], and $\mathbf{Irr}$ is a two sided ideal of $Virt(m_0 + m_1, n)$.*

Definition 3.6. ***(The Capelli devirtualization epimorphism)*** *The Capelli epimorphism is the projection*

$$\mathfrak{p} : Virt(m_0 + m_1, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \twoheadrightarrow \mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$$

with $Ker(\mathfrak{p}) = \mathbf{Irr}$.

Remark 3.7. *Any element in $Virt(m_0+m_1, n)$ defines an element in $\mathbf{U}(gl(n))$, and is called a virtual presentation of it. The map $\mathfrak{p}$ being a surjection, any element $\mathbf{p} \in \mathbf{U}(gl(n))$ admits several virtual presentations. We even take virtual presentations as definitions of special elements in $\mathbf{U}(gl(n))$, and this method will turn out to be quite effective.*

The next results will play a crucial role in Section 4.

Proposition 3.8. *For every $e_{x_i,x_j} \in gl(n) \subset gl(m_0|m_1+n)$, let $ad(e_{x_i,x_j})$ denote its adjoint action on $Virt(m_0+m_1,n)$; the ideal* **Irr** *is $ad(e_{x_i,x_j})-$invariant. Then*

$$\mathfrak{p}\left(ad(e_{x_i,x_j})(\mathbf{m})\right) = ad(e_{x_i,x_j})\left(\mathfrak{p}(\mathbf{m})\right), \qquad \mathbf{m} \in Virt(m_0+m_1,n). \tag{3}$$

Corollary 3.9. *The Capelli epimorphism image of an element of $Virt(m_0|m_1+n)$ that is an invariant for the adjoint action of $gl(n)$ is in the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$.*

Example 3.10. (A virtual presentation of the Capelli determinant) Let $\alpha \in A_0$. The element

$$M = e_{x_n,\alpha} \cdots e_{x_2,\alpha} e_{x_1,\alpha} \cdot e_{\alpha,x_1} e_{\alpha,x_2} \cdots e_{\alpha,x_n} \tag{4}$$

belongs to the subalgebra $Virt(m_0|m_1+n)$ (section 3.5.3 below). The image of the element M under the Capelli devirtualization epimorphism $\mathfrak{p}$ equals the *column determinant*[3]

$$\mathbf{H}_n(n) = \mathbf{cdet}\begin{pmatrix} e_{x_1,x_1}+(n-1) & e_{x_1,x_2} & \dots & e_{x_1,x_n} \\ e_{x_2,x_1} & e_{x_2,x_2}+(n-2) & \dots & e_{x_2,x_n} \\ \vdots & \vdots & \vdots & \\ e_{x_n,x_1} & e_{x_n,x_2} & \dots & e_{x_n,x_n} \end{pmatrix}.$$

This result is a special case of the result that we called the "Laplace expansion for Capelli rows" ([19] Theorem 2, [11] Theorem 6.3). A sketchy proof of it can also be found in [46].

Recall that

$$ad(e_{x_i,x_j})\left(e_{x_h,\alpha}\right) = \delta_{jh} e_{x_i,\alpha},$$
$$ad(e_{x_i,x_j})\left(e_{\alpha,x_k}\right) = -\delta_{ki} e_{\alpha,x_j},$$

for every virtual symbol α, and that $ad(e_{x_i,x_j})$ acts as a derivation, for every $i,j = 1,2,\dots,n$.

The monomial M is annihilated by $ad(e_{x_i,x_j})$, $i \neq j$, by skew-symmetry. Furthermore, $ad(e_{x_i,x_i})(M) = M - M = 0$, $i = 1,2,\dots,n$; hence, M is an invariant for the adjoint action of $gl(n)$.

Since $\mathfrak{p}(M) = \mathbf{H}_n(n)$, the element $\mathbf{H}_n(n)$ is central in $\mathbf{U}(gl(n))$, by Corollary 3.9.

3.5.2 The action of $Virt(m_0+m_1,n)$ on the subalgebra $\mathbb{C}[M_{n,d}]$

From the representation-theoretic point of view, the core of the *method of virtual variables* lies in the following result.

[3] The symbol **cdet** denotes the column determinat of a matrix $A = [a_{ij}]$ with noncommutative entries: $\mathbf{cdet}(A) = \sum_\sigma (-1)^{|\sigma|}\ a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.

Theorem 3.11. *The action of $Virt(m_0 + m_1, n)$ leaves invariant the subalgebra of algebraic forms $\mathbb{C}[M_{n,d}] \subseteq \mathbb{C}[M_{m_0|m_1+n,d}]$, and, therefore, the action of $Virt(m_0 + m_1, n)$ on $\mathbb{C}[M_{n,d}]$ is well defined. Furthermore, for every $\mathbf{v} \in Virt(m_0 + m_1, n)$, its action on $\mathbb{C}[M_{n,d}]$ equals the action of $\mathfrak{p}(\mathbf{v})$.*

Therefore, instead of studying the action of an element in $\mathbf{U}(gl(n))$, one can study the action of a virtual presentation of it in $Virt(m_0|m_1 + n)$. The advantage of virtual presentations is that they are frequently of monomial form, admit quite transparent interpretations and are much easier to be dealt with (see, e.g. [14], [15], [19], [11], [12]).

A prototypical instance of this method is provided by the celebrated Capelli identity [21], [71], [39], [40], [68]. From Example 3.10, it follows that the action of the Capelli determinant $\mathbf{H}_n(n)$ on a form $f \in \mathbb{C}[M_{n,d}]$ is the same as the action of its monomial virtual presentation (4), and this leads to a few lines proof of the identity [19], [12].

3.5.3 Balanced monomials as elements of the virtual algebra $Virt(m_0+m_1, n)$

In order to make the virtual variables method effective, we need to exhibit a class of nontrivial elements that belong to $Virt(m_0 + m_1, n)$. A quite relevant class of such elements is provided by *balanced monomials*.

Definition 3.12. ***(Balanced monomials)*** *In the algebra $\mathbf{U}(gl(m_0|m_1 + n))$, consider an element of the forms:*

- $e_{x_{i_1},\gamma_{p_1}} \cdots e_{x_{i_k},\gamma_{p_k}} \cdot e_{\gamma_{p_1},x_{j_1}} \cdots e_{\gamma_{p_k},x_{j_k}}$,
- $e_{x_{i_1},\theta_{q_1}} \cdots e_{x_{i_k},\theta_{q_k}} \cdot e_{\theta_{q_1},\gamma_{p_1}} \cdots e_{\theta_{q_k},\gamma_{p_k}} \cdot e_{\gamma_{p_1},x_{j_1}} \cdots e_{\gamma_{p_k},x_{j_k}}$,
- $\cdots\cdots \quad \cdots\cdots$

where $x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_k} \in L$, i.e., the $x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_k}$ are k proper symbols;

In plain words, a balanced monomial is product of two or more factors where the rightmost one *annihilates* the k proper symbols $x_{j_1}, \ldots, x_{j_k}$ and *creates* some virtual symbols; the leftmost one *annihilates* all the virtual symbols and *creates* the k proper symbols $x_{i_1}, \ldots, x_{i_k}$; between these two factors, there might be further factors that annihilate and create virtual symbols only.

The next result is the (superalgebraic) formalization of the argument developed by Capelli in [25], CAPITOLO I, §X.Metodo delle variabili ausiliarie, page 55 ff.

Proposition 3.13. *([14], [15], [19], [11], [12]) Every balanced monomial belongs to $Virt(m_0 + m_1, n)$. Hence its image under the Capelli epimorphism $\mathfrak{p}$ belongs to $\mathbf{U}(gl(n))$.*

In plain words, the action of a balanced monomial on the subalgebra $\mathbb{C}[M_{n,d}]$ equals the action of a suitable element of $\mathbf{U}(gl(n))$.

The following result lies deeper and is a major tool in the proof of identities involving monomial virtual presentation of elements of $\mathbf{U}(gl(n))$. Since the adjoint representation acts by superderivation, it may be regarded as a version of the *Laplace expansion* for the images of balanced monomials.

Proposition 3.14. ***(Monomial virtual presentation and adjoint actions)*** *In* $\mathbf{U}(gl(n))$, *the element*

$$\mathfrak{p}\left[e_{x_{i_1},\gamma_{p_1}} \cdots e_{x_{i_n},\gamma_{p_n}} \cdot e_{\gamma_{p_1},x_{j_1}} \cdots e_{\gamma_{p_n},x_{j_n}}\right]$$

equals

$$\mathfrak{p}\Big[\ ad(e_{x_{i_1},\gamma_{p_1}}) \cdots ad(e_{x_{i_n},\gamma_{p_n}})\big(e_{\gamma_{p_1},x_{j_1}} e_{\gamma_{p_2},x_{j_2}} \cdots e_{\gamma_{p_n},x_{j_n}}\big)\ \Big].$$

Example 3.15. Let $\alpha \in A_0$. Then

$$[x_{i_k} \cdots x_{i_2} x_{i_1} | x_{i_1} x_{i_2} \cdots x_{i_k}] = \mathfrak{p}[e_{x_{i_k},\alpha} \cdots e_{x_{i_2},\alpha} e_{x_{i_1},\alpha} \cdot e_{\alpha,x_{i_1}} e_{\alpha,x_{i_2}} \cdots e_{\alpha,x_{i_k}}] =$$

$$= \mathfrak{p}\Big[\ ad(e_{x_{i_k},\alpha}) \cdots ad(e_{x_{i_2},\alpha}) ad(e_{x_{i_1},\alpha}) \Big(e_{\alpha,x_{i_1}} e_{\alpha,x_{i_2}} \cdots e_{\alpha,x_{i_k}}\Big)\ \Big] =$$

$$= \mathbf{cdet}\begin{pmatrix} e_{x_{i_1},x_{i_1}} + (k-1) & e_{x_{i_1},x_{i_2}} & \ldots & e_{x_{i_1},x_{i_k}} \\ e_{x_{i_2},x_{i_1}} & e_{x_{i_2},x_{i_2}} + (k-2) & \ldots & e_{x_{i_2},x_{i_k}} \\ \vdots & \vdots & \vdots & \\ e_{x_{i_k},x_{i_1}} & e_{x_{i_k},x_{i_2}} & \ldots & e_{x_{i_k},x_{i_k}} \end{pmatrix} \in \mathbf{U}(gl(n)).$$

Example 3.16. Let $\alpha \in A_1$. The element

$$\mathbf{p} = \mathfrak{p}\left[e_{x_3,\alpha} e_{x_2,\alpha} e_{x_1,\alpha} \cdot e_{\alpha,x_1} e_{\alpha,x_2} e_{\alpha,x_3}\right] =$$

$$= \mathfrak{p}\left[ad(e_{x_3,\alpha}) ad(e_{x_2,\alpha}) ad(e_{x_1,\alpha}) \left(e_{\alpha,x_1} e_{\alpha,x_2} e_{\alpha,x_3}\right)\right]$$

equals the *column permanent*[4]

$$\mathbf{cper}\begin{pmatrix} e_{x_1,x_1} - 2 & e_{x_1,x_2} & e_{x_1,x_3} \\ e_{x_2,x_1} & e_{x_2,x_2} - 1 & e_{x_2,x_3} \\ e_{x_3,x_1} & e_{x_3,x_2} & e_{x_3,x_3} \end{pmatrix} \in \mathbf{U}(gl(3)).$$

Example 3.17. Let $\alpha, \beta \in A_0$. Then

$$\left[\begin{array}{c|c} x_1 & x_2 \\ x_2 & x_1 \end{array}\right] = \mathfrak{p}\left(e_{x_1,\alpha} e_{x_2,\beta} \cdot e_{\alpha,x_2} e_{\beta,x_1}\right) =$$

$$= -e_{x_1,x_2} e_{x_2,x_1} + e_{x_1,x_1} \in \mathbf{U}(gl(2)).$$

[4]The symbol **cper** denotes the column permanent of a matrix $A = [a_{ij}]$ with noncommutative entries: $\mathbf{cper}(A) = \sum_\sigma\ a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.

3.6 Bitableaux in $\mathbf{U}(gl(m_0|m_1{+}n))$ and balanced monomials

3.6.1 Young tableaux on the alphabet $A_0 \cup A_1 \cup L$

Consider the alphabets A_0, A_1, L, (that is a distinguished $\mathbb{Z}_2-$homogeneous bases of the $\mathbb{Z}_2-$graded vector space $W = W_0 \oplus W_1$, $W_0 = V_{m_0}$, $W_1 = V_{m_1} \oplus V_n$), where $A_0 = \{\alpha_1, \ldots, \alpha_{m_0}\}$, $A_1 = \{\beta_1, \ldots, \beta_{m_1}\}$, $L = \{x_1, \ldots, x_n\}$ denote distinguished bases of V_{m_0}, V_{m_1} and V_n.

Given a partition λ, a *Young tableau* X of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ on the alphabet $A_0 \cup A_1 \cup L$ is an array

$$X = \left(\begin{array}{lll} z_{i_1} \cdots\cdots\cdots & & z_{i_{\lambda_1}} \\ z_{j_1} \cdots\cdots z_{j_{\lambda_2}} & & \\ \cdots\cdots & & \\ z_{s_1} \ldots z_{s_{\lambda_p}} & & \end{array} \right),$$

where the symbols z's belong to the alphabet $A_0 \cup A_1 \cup L$. The partition λ is called the *shape* of the a Young tableau X, denoted by the symbol $sh(X)$.

We also denote the Young tableau X by the sequence of its *row words*, namely

$$X = (\omega_1, \omega_2, \ldots, \omega_p),$$

where

$$\omega_k = z_{k_1} \ldots z_{k_{\lambda_k}}, \quad k = 1, 2, \ldots, p.$$

The *row word* of the Young tableau X is the word

$$\omega(X) = \omega_1 \omega_2 \cdots \omega_p. \tag{5}$$

The *content* of the Young tableau X is the function

$$c_X : A_0 \cup A_1 \cup L \to \mathbb{N},$$

where $c_X(z)$ equals the number of occurrences of z in X, for $z \in A_0 \cup A_1 \cup L$.

Consider the linear order

$$\alpha_1 < \ldots < \alpha_{m_0} < \beta_1 < \ldots < \beta_{m_1} < x_1 < \ldots < x_n$$

on the alphabet $A_0 \cup A_1 \cup L$. The Young tableau X is said to be *(super)standard* whenever its row and column are nondecreasing words, its rows have no repetitions of negative symbols in $A_1 \cup L$, its columns no repetitions of positive symbols in A_0. ([6], [7], [38], [14]).

3.6.2 Bitableaux monomials

Let S and T be two Young tableaux of same shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ on the alphabet $A_0 \cup A_1 \cup L$:

$$S = \begin{pmatrix} z_{i_1} \dots\dots\dots & & z_{i_{\lambda_1}} \\ z_{j_1} \dots\dots z_{j_{\lambda_2}} & & \\ \dots\dots & & \\ z_{s_1} \dots z_{s_{\lambda_p}} & & \end{pmatrix}, \qquad T = \begin{pmatrix} z_{h_1} \dots\dots\dots & & z_{h_{\lambda_1}} \\ z_{k_1} \dots\dots z_{k_{\lambda_2}} & & \\ \dots\dots & & \\ z_{t_1} \dots z_{t_{\lambda_p}} & & \end{pmatrix} \tag{6}$$

.

To the pair (S, T), we associate the *bitableau monomial*:

$$e_{S,T} = e_{z_{i_1}, z_{h_1}} \cdots e_{z_{i_{\lambda_1}}, z_{h_{\lambda_1}}} e_{z_{j_1}, z_{k_1}} \cdots e_{z_{j_{\lambda_2}}, z_{k_{\lambda_2}}} \cdots\cdots e_{z_{s_1}, z_{t_1}} \cdots e_{z_{s_{\lambda_p}}, z_{t_{\lambda_p}}}$$

in $\mathbf{U}(gl(m_0|m_1 + n))$.

Example 3.18. Let S and T be tableaux of shape $\lambda = (3, 2, 2)$ on the alphabet $A_0 \cup A_1 \cup L$:

$$S = \begin{pmatrix} z & x & y \\ z & u & \\ x & v & \end{pmatrix}, \qquad T = \begin{pmatrix} z & s & w \\ x & t & \\ y & w & \end{pmatrix} \tag{7}$$

. To the pair (S, T), we associate the monomial:

$$e_{S,T} = e_{z,z} e_{x,s} e_{y,w} e_{z,x} e_{u,t} e_{x,y} e_{v,w}$$

in $\mathbf{U}(gl(m_0|m_1 + n))$.

3.6.3 Deruyts and Coderuyts tableaux

Let $\alpha_1, \dots, \alpha_p \in A_0$, $\beta_1, \dots, \beta_{\lambda_1} \in A_1$ and set

$$D^*_\lambda = \begin{pmatrix} \beta_1 \dots\dots\dots & & \beta_{\lambda_1} \\ \beta_1 \dots\dots \beta_{\lambda_2} & & \\ \dots\dots & & \\ \beta_1 \dots \beta_{\lambda_p} & & \end{pmatrix}, \qquad C^*_\lambda = \begin{pmatrix} \alpha_1 \dots\dots\dots & & \alpha_1 \\ \alpha_2 \dots\dots \alpha_2 & & \\ \dots\dots & & \\ \alpha_p \dots \alpha_p & & \end{pmatrix} \tag{8}$$

.

The tableaux D^*_λ and C^*_λ are called the *virtual Deruyts and Coderuyts tableaux* of shape λ, respectively.

3.6.4 Three special classes of elements in $Virt(m_0 + m_1, n)$

Given a pair (S, T) of Young tableaux of the same shape λ on the proper alphabet L, consider the elements

$$e_{S,C^*_\lambda}\ e_{C^*_\lambda,T} \in \mathbf{U}(gl(m_0|m_1 + n)), \tag{9}$$

$$e_{S,D^*_\lambda}\ e_{D^*_\lambda,T} \in \mathbf{U}(gl(m_0|m_1 + n)), \tag{10}$$

$$e_{S,D^*_\lambda}\ e_{D^*_\lambda,C^*_\lambda}\ e_{C^*_\lambda,T} \in \mathbf{U}(gl(m_0|m_1 + n)). \tag{11}$$

Since elements (9), (10), (11) are balanced monomials in $\mathbf{U}(gl(m_0|m_1+n))$, then they belong to the subalgebra $Virt(m_0+m_1, n)$ (section 3.5.3). Hence, we can consider their images with respect to the Capelli epimorphism $\mathfrak{p}$ and set

$$\mathfrak{p}\Big(e_{S,C^*_\lambda}\ e_{C^*_\lambda,T}\Big) = SC^*_\lambda\ C^*_\lambda T \in \mathbf{U}(gl(n)), \tag{12}$$

$$\mathfrak{p}\Big(e_{S,D^*_\lambda}\ e_{D^*_\lambda,T}\Big) = SD^*_\lambda\ D^*_\lambda T \in \mathbf{U}(gl(n)), \tag{13}$$

$$\mathfrak{p}\Big(e_{S,D^*_\lambda}\ e_{D^*_\lambda,C^*_\lambda}\ e_{C^*_\lambda,T}\Big) = SD^*_\lambda\ D^*_\lambda C^*_\lambda\ C^*_\lambda T \in \mathbf{U}(gl(n)). \tag{14}$$

Example 3.19. Let $\lambda = (3,2,2)$, then

$$C^*_\lambda = \left(\begin{array}{ccc} \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \\ \alpha_3 & \alpha_3 & \end{array}\right), \quad \alpha_1, \alpha_2, \alpha_3 \in A_0. \tag{15}$$

Let S and T be the tableaux of shape $\lambda = (3,2,2)$ on the alphabet $A_0 \cup A_1 \cup L$, of Example 3.18. Then

$$SC^*_\lambda\ C^*_\lambda T = \mathfrak{p}\Big(e_{S,C^*_\lambda}\ e_{C^*_\lambda,T}\Big) = \mathfrak{p}\Big(e_{\substack{xyx \\ zu \\ xv}\ \substack{\alpha_1\alpha_1\alpha_1 \\ \alpha_2\alpha_2 \\ \alpha_3\alpha_3}}\ \ e_{\substack{\alpha_1\alpha_1\alpha_1 \\ \alpha_2\alpha_2 \\ \alpha_3\alpha_3}\ \substack{zsw \\ xt \\ yw}}\Big) =$$

$$= \mathfrak{p}\Big(e_{z,\alpha_1}e_{x,\alpha_1}e_{y,\alpha_1}e_{z,\alpha_2}e_{u,\alpha_2}e_{x,\alpha_3}e_{v,\alpha_3}e_{\alpha_1,z}e_{\alpha_1,s}e_{\alpha_1,w}e_{\alpha_2,x}e_{\alpha_2,t}e_{\alpha_3,y}e_{\alpha_3,w}\Big).$$

Remark 3.20. *In the present notation, the element described in Example 3.15 is the element*

$$(-1)^{\binom{k}{2}}\ SC^*_\lambda\ C^*_\lambda S, \quad \lambda = (k), \quad S = \left(\ x_{i_1} \quad \ldots \quad x_{i_k}\ \right).$$

The element described in Example 3.16 is the element

$$SD^*_\lambda\ D^*_\lambda S, \quad \lambda = (1,1,1), \quad S = \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right).$$

The element described in Example 3.17 is the element

$$SC^*_\lambda\ C^*_\lambda T, \quad \lambda = (1,1), \quad S = \left(\begin{array}{c} x_1 \\ x_2 \end{array}\right), \quad T = \left(\begin{array}{c} x_2 \\ x_1 \end{array}\right)$$

From Proposition 3.8, it directly follows:

Proposition 3.21. *For every $e_{x_i,x_j} \in gl(n) \subset gl(m_0|m_1+n)$, let $ad(e_{x_i,x_j})$ denote its adjoint action on $Virt(m_0+m_1,n)$. We have*

$$ad(e_{x_i,x_j})\,(SC^*_\lambda\ C^*_\lambda T) = \mathfrak{p}\Big(ad(e_{x_i,x_j})\,\big(e_{S,C^*_\lambda}\ e_{C^*_\lambda,T}\big)\Big) \in \mathbf{U}(gl(n)), \tag{16}$$

$$ad(e_{x_i,x_j})\,(SD^*_\lambda\ D^*_\lambda T) = \mathfrak{p}\Big(ad(e_{x_i,x_j})\,\big(e_{S,D^*_\lambda}\ e_{D^*_\lambda,T}\big)\Big) \in \mathbf{U}(gl(n)), \tag{17}$$

$$ad(e_{x_i,x_j})\,(SD^*_\lambda\ D^*_\lambda C^*_\lambda\ C^*_\lambda T) = \mathfrak{p}\Big(ad(e_{x_i,x_j})\,\big(e_{S,D^*_\lambda}\ e_{D^*_\lambda,C^*_\lambda}\ e_{C^*_\lambda,T}\big)\Big) \in \mathbf{U}(gl(n)). \tag{18}$$

3.7 Bitableaux in $\mathbb{C}[M_{m_0|m_1+n,d}]$ and the standard monomial theory

3.7.1 Biproducts in $\mathbb{C}[M_{m_0|m_1+n,d}]$

Embed the algebra

$$\mathbb{C}[M_{m_0|m_1+n,d}] = \mathbb{C}[(\alpha_s|j),(\beta_t|j),(x_i|j)]$$

into the (supersymmetric) algebra $\mathbb{C}[(\alpha_s|j),(\beta_t|j),(x_i|j),(\gamma|j)]$ generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j),(\beta_t|j),(x_i|j),(\gamma|j)$, $j = 1,2,\ldots,d$, where

$$|(\gamma|j)| = 1 \in \mathbb{Z}_2 \quad for\ every\ j = 1,2,\ldots,d,$$

and denote by $D_{z_i,\gamma}$ the superpolarization of γ to z_i.

Definition 3.22. ***(Biproducts in*** $\mathbb{C}[M_{m_0|m_1+n,d}]$***)*** *Let $\omega = z_1z_2\cdots z_p$ be a word on $A_0 \cup A_1 \cup L$, and $\varpi = j_{t_1}j_{t_2}\cdots j_{t_q}$ a word on the alphabet $P = \{1,2,\ldots,d\}$. The biproduct*

$$(\omega|\varpi) = (z_1z_2\cdots z_p|j_{t_1}j_{t_2}\cdots j_{t_q})$$

is the element

$$D_{z_1,\gamma}D_{z_2,\gamma}\cdots D_{z_p,\gamma}\Big((\gamma|j_{t_1})(\gamma|j_{t_2})\cdots(\gamma|j_{t_q})\Big) \in \mathbb{C}[M_{m_0|m_1+n,d}]$$

if $p = q$ and is set to be zero otherwise.

Claim 3.23. *The biproduct $(\omega|\varpi) = (z_1z_2\cdots z_p|j_{t_1}j_{t_2}\cdots j_{t_q})$ is supersymmetric in the z's and skew-symmetric in the j's. In symbols*

1. $(z_1z_2\cdots z_iz_{i+1}\cdots z_p|j_{t_1}j_{t_2}\cdots j_{t_q}) = (-1)^{|z_i||z_{i+1}|}(z_1z_2\cdots z_{i+1}z_i\cdots z_p|j_{t_1}j_{t_2}\cdots j_{t_q})$
2. $(z_1z_2\cdots z_iz_{i+1}\cdots z_p|j_{t_1}j_{t_2}\cdots j_{t_i}j_{t_{i+1}}\cdots j_{t_q}) = -(z_1z_2\cdots z_iz_{i+1}\cdots z_p|j_{t_1}\cdots j_{t_{i+1}}j_{t_i}\cdots j_{t_q})$.

Proposition 3.24. ***(Laplace expansions)*** *We have*

1. $(\omega_1\omega_2|\varpi) = \Sigma_{(\varpi)}\ (-1)^{|\varpi_{(1)}||\omega_2|}\ (\omega_1|\varpi_{(1)})(\omega_2|\varpi_{(2)})$.
2. $(\omega|\varpi_1\varpi_2) = \Sigma_{(\omega)}\ (-1)^{|\varpi_1||\omega_{(2)}|}\ (\omega_{(1)}|\varpi_1)(\omega_{(2)}|\varpi_2.)$

where

$$\triangle(\varpi) = \Sigma_{(\varpi)}\ \varpi_{(1)} \otimes \varpi_{(2)}, \quad \triangle(\omega) = \Sigma_{(\omega)}\ \omega_{(1)} \otimes \omega_{(2)}$$

denote the coproducts in the Sweedler notation (see, e.g [1]) of the elements ϖ and ω in the supersymmetric Hopf algebra of W (see, e.g [11]) and in the free exterior Hopf algebra generated by $j = 1,2,\ldots,d$, respectively.

Example 3.25. Let $\omega = x_1x_2x_3$, $\varpi = 123$. Then

$$(\omega|\varpi) = D_{x_1,\gamma}D_{x_2,\gamma}D_{x_3,\gamma}\big((\gamma|1)(\gamma|2)(\gamma|3)\big) = -det\big[(x_i|j)\big]_{i,j=1,2,3},$$

where the $(x_i|j)$'s are commutative variables.

Example 3.26. Let $\omega = \alpha_1\alpha_2x_3$, $\varpi = 123$, where $|(\alpha_1|j)| = |(\alpha_2|j)| = 1$, $j = 1, 2, 3$ and $|(x_3|j)| = 0$, $j = 1, 2, 3$. Then

$$\begin{aligned}
(\omega|\varpi) &= D_{\alpha_1,\gamma}D_{\alpha_2,\gamma}D_{x_3,\gamma}\big((\gamma|1)(\gamma|2)(\gamma|3)\big) \\
&= D_{\alpha_1,\gamma}D_{\alpha_2,\gamma}\Big((x_3|1)(\gamma|2)(\gamma|3) - (\gamma|1)(x_3|2)(\gamma|3) + (\gamma|1)(\gamma|2)(x_3|3)\Big) \\
&= D_{\alpha_1,\gamma}\Big((x_3|1)(\alpha_2|2)(\gamma|3) + (x_3|1)(\gamma|2)(\alpha_2|3) - (\alpha_2|1)(x_3|2)(\gamma|3) \\
&\qquad - (\gamma|1)(x_3|2)(\alpha_2|3) + (\alpha_2|1)(\gamma|2)(x_3|3) + (\gamma|1)(\alpha_2|2)(x_3|3)\Big) \\
&= (x_3|1)(\alpha_2|2)(\alpha_1|3) + (x_3|1)(\alpha_1|2)(\alpha_2|3) - (\alpha_2|1)(x_3|2)(\alpha_1|3) \\
&\quad - (\alpha_1|1)(x_3|2)(\alpha_2|3) + (\alpha_2|1)(\alpha_1|2)(x_3|3) + (\alpha_1|1)(\alpha_2|2)(x_3|3).
\end{aligned}$$

From Proposition 3.24.1, by setting $\varpi_1 = 12$, $\varpi_2 = 3$, it follows

$$(\omega|\varpi) = (\alpha_1\alpha_2|12)(x_3|3) + (\alpha_1x_3|12)(\alpha_2|3) + (\alpha_2x_3|12)(\alpha_1|3).$$

From Proposition 3.24.2, by setting $\omega_1 = \alpha_1\alpha_2$, $\omega_2 = x_3$, it follows

$$(\omega|\varpi) = (\alpha_1\alpha_2|12)(x_3|3) - (\alpha_1\alpha_2|13)(x_3|2) + (\alpha_1\alpha_2|23)(x_3|1).$$

3.7.2 Biproducts and polarization operators

Following the notation introduced in the previous sections, let

$$Super[W] = Sym[W_0] \otimes \Lambda[W_1]$$

denote the *(super)symmetric* algebra of the space

$$W = W_0 \oplus W_1$$

(see, e.g. [65], [70]).

By multilinearity, the algebra $Super[W]$ is the same as the superalgebra $Super[A_0 \cup A_1 \cup L]$ generated by the "variables"

$$\alpha_1, \ldots, \alpha_{m_0} \in A_0, \quad \beta_1, \ldots, \beta_{m_1} \in A_1, \quad x_1, \ldots, x_n \in L,$$

modulo the congruences

$$zz' = (-1)^{|z||z'|}z'z, \quad z, z' \in A_0 \cup A_1 \cup L.$$

Let $d_{z,z'}$ denote the polarization operator of z' to z on

$$Super[W] = Super[A_0 \cup A_1 \cup L],$$

that is the unique superderivation of $\mathbb{Z}_2$-degree

$$|z| + |z'| \in \mathbb{Z}_2$$

such that

$$d_{z,z'}(z'') = \delta_{z',z''} \cdot z,$$

for every $z, z', z'' \in A_0 \cup A_1 \cup L$.

Clearly, the map

$$e_{z,z'} \to d_{z,z'}$$

is a Lie superalgebra map and, therefore, induces a structure of

$$gl(m_0|m_1 + n) - module$$

on $Super[A_0 \cup A_1 \cup L] = Super[W]$.

Proposition 3.27. *Let* $\varpi = j_{t_1} j_{t_2} \cdots j_{t_q}$ *be a word on* $P = \{1, 2, \ldots, d\}$*. The map*

$$\Phi_\varpi : \omega \mapsto (\omega|\varpi),$$

ω any word on $A_0 \cup A_1 \cup L$, uniquely defines $gl(m_0|m_1 + n)-$equivariant linear operator

$$\Phi_\varpi : Super[A_0 \cup A_1 \cup L] \to \mathbb{C}[M_{m_0|m_1+n,d}],$$

that is

$$\Phi_\varpi\big(d_{z,z'}(\omega)\big) = D_{z,z'}\big((\omega|\varpi)\big), \tag{19}$$

for every $z, z' \in A_0 \cup A_1 \cup L$.

With a slight abuse of notation, we will write (19) in the form

$$D_{z,z'}\big((\omega|\varpi)\big) = (d_{z,z'}(\omega)|\varpi). \tag{20}$$

3.7.3 Bitableaux in $\mathbb{C}[M_{m_0|m_1+n,d}]$

Let $S = (\omega_1, \omega_2, \ldots, \omega_p$ and $T = (\varpi_1, \varpi_2, \ldots, \varpi_p)$ be Young tableaux on $A_0 \cup A_1 \cup L$ and $P = \{1, 2, \ldots, d\}$ of shapes λ and μ, respectively.

If $\lambda = \mu$, the *Young bitableau* $(S|T)$ is the element of $\mathbb{C}[M_{m_0|m_1+n,d}]$ defined as follows:

$$(S|T) = \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array}\right) = \pm\ (\omega_1)|\varpi_1)(\omega_2)|\varpi_2)\cdots(\omega_p)|\varpi_p),$$

where

$$\pm = (-1)^{|\omega_2||\varpi_1| + |\omega_3|(|\varpi_1| + |\varpi_2|) + \cdots + |\omega_p|(|\varpi_1| + |\varpi_2| + \cdots + |\varpi_{p-1}|)}.$$

If $\lambda \neq \mu$, the *Young bitableau* $(S|T)$ is set to be zero.

3.7.4 Bitableaux and polarization operators

By naturally extending the slight abuse of notation (20), the action of any polarization on bitableaux can be explicitly described:

Proposition 3.28. *Let $z, z' \in A_0 \cup A_1 \cup L$, and let $S = (\omega_1, \ldots, \omega_p)$, $T = (\varpi_1, \ldots, \varpi_p)$. We have the following identity:*

$$
\begin{aligned}
D_{z,z'}(S\,|\,T) \ &= \ D_{z,z'} \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array} \right) \\
&= \sum_{s=1}^{p} (-1)^{(|z|+|z'|)\epsilon_s} \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ d_{z,z'}(\omega_s) & \vdots \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array} \right),
\end{aligned}
$$

where

$$\epsilon_1 = 1, \quad \epsilon_s = |\omega_1| + \cdots + |\omega_{s-1}|, \quad s = 2, \ldots, p.$$

Example 3.29. Let $\alpha_i \in A_0$, $x_1, x_2, x_3, x_4 \in L$, $|D_{\alpha_i, x_2}| = 1$. Then

$$D_{\alpha_i, x_2} \left(\begin{array}{l|l} x_1\ x_3\ x_2 & 1\ 2\ 3 \\ x_2\ x_3 & 2\ 3 \\ x_4\ x_2 & 3\ 1 \end{array} \right) =$$

$$= \left(\begin{array}{l|l} x_1\ x_3\ \alpha_i & 1\ 2\ 3 \\ x_2\ x_3 & 2\ 3 \\ x_4\ x_2 & 3\ 1 \end{array} \right) - \left(\begin{array}{l|l} x_1\ x_3\ x_2 & 1\ 2\ 3 \\ \alpha_i\ x_3 & 2\ 3 \\ x_4\ x_2 & 3\ 1 \end{array} \right) + \left(\begin{array}{l|l} x_1\ x_3\ x_2 & 1\ 2\ 3 \\ x_2\ x_3 & 2\ 3 \\ x_4\ \alpha_i & 3\ 1 \end{array} \right)$$

3.7.5 The straightening algorithm and the standard basis theorem for $\mathbb{C}[M_{m_0|m_1+n,d}]$

Consider the set of all bitableaux $(S|T) \in \mathbb{C}[M_{m_0|m_1+n,d}]$, where $sh(S) = sh(T) \vdash h$, h a given positive integer. In the following, let denote by $\leq$ the partial order on this set defined by the following two steps:

- $(S|T) < (S'|T')$ whenever $sh(S) <_l sh(S')$,
- $(S|T) < (S'|T')$ whenever $sh(S) = sh(S')$, $w(S) >_l w(S')$, $w(T) >_l w(T')$,

where the shapes and the row-words are compared in the lexicographic order.

The next results are superalgebraic versions of classical, well-known results for the symmetric algebra $\mathbb{C}[M_{n,d}]$ ([33], [29], [28], for the general theory of standard monomials see, e.g. [60], Chapt. 13) and of their skew-symmetric analogues ([32], [3]).

Theorem 3.30. *(The straightening algorithm) [38]*

Let $(P|Q) \in \mathbb{C}[M_{m_0|m_1+n,d}]$. Then $(P|Q)$ can be written as a linear combination, with rational coefficients,

$$(P|Q) = \sum_{S,T} c_{S,T}\ (S|T), \tag{21}$$

of standard bitableaux $(S|T)$, where $(S|T) \geq (P|Q)$ and $c_S = c_P$, $c_T = c_Q$.

Since standard bitableaux are linearly independent in $\mathbb{C}[M_{m_0|m_1+n,d}]$, the expansion (21) is unique.

Following [6], [7], [14], a partition λ satisfies the $(m_0, m_1+n)-$*hook condition* (in symbols, $\lambda \in H(m_0, m_1+n)$) if and only if $\lambda_{m_0+1} \leq m_1 + n$. We have:

Lemma 3.31. *There exists a standard tableau on $A_0 \cup A_1 \cup L$ of shape λ if and only if $\lambda \in H(m_0, m_1+n)$.*

Given a positive integer $h \in \mathbb{Z}^+$, let $\mathbb{C}_h[M_{m_0|m_1+n,d}]$ denote the $h-$th homogeneous component of $\mathbb{C}[M_{m_0|m_1+n,d}]$.

From Theorem 3.30, it follows

Corollary 3.32. *(The Standard basis theorem for $\mathbb{C}_h[M_{m_0|m_1+n,d}]$, [38])*

The following set is a basis of $\mathbb{C}_h[M_{m_0|m_1+n,d}]$:

$$\{(S|T)\ standard;\ sh(S) = sh(T) = \lambda \vdash h, \lambda \in H(m_0, m_1+n), \lambda_1 \leq d\ \}.$$

3.8 The Schur (covariant) $\mathbf{U}(gl(m_0|m_1+n))-$supermodules

3.8.1 The Schur supermodules as submodules of $\mathbb{C}[M_{m_0|m_1+n,d}]$

In the following, we regard the superalgebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ as $\mathbf{U}(gl(m_0|m_1+n))-$module. We recall the definition of a "Schur supermodule" and the main facts about them. This material goes back to our work of $1988-1989$ ([14], [15]); our construction produces irreducible $gl(m_0|m_1+n))-$supermodules that are isomorphic to the modules constructed by Berele and Regev [6], [7] as tensor modules induced by Young symmetrizers (see, e.g. [71]) when they act by a "signed action" of the symmetric group (see also [31], [42]). The description presented here is simpler than the tensor description, provides a close connection with the superstraightening theory of Grosshans, Rota and Stein [38], and allows the action of $\mathbf{U}(gl(m_0|m_1+n))$ to be described in a transparent way (Proposition 3.28).

Given $\lambda \in H(m_0, m_1+n)$, the *Schur supermodule* $Schur_\lambda(m_0, m_1+n)$ is the subspace of $\mathbb{C}[M_{m_0|m_1+n,d}]$, $d \geq \lambda_1$, spanned by the set of all bitableaux $(S|D^P_\lambda)$ of shape λ, where D^P_λ is the *Deruyts* tableau on $P = \{1, 2, \ldots, d\}$

$$D^P_\lambda = \left(\begin{array}{l} 1 \ldots\ldots\ldots\ \ \lambda_1 \\ 1 \ldots\ldots \lambda_2 \\ \ldots\ldots \\ 1 \ldots \lambda_p \end{array} \right), \quad p = l(\lambda).$$

and S is a Young tableau on the alphabet $A_0 \cup A_1 \cup L$.

From Theorem 3.30 and Corollary 3.32, it follows

Proposition 3.33. *The set*

$$\Big\{(S|D_\lambda^P); S \ \textit{superstandard}\ \Big\}$$

is a $\mathbb{C}-$*linear basis of* $Schur_\lambda(m_0, m_1 + n)$.

Furthermore, we recall

Proposition 3.34. *([14], [11]) The submodule* $Schur_\lambda(m_0, m_1 + n)$ *is an irreducible* $\mathbf{U}(gl(m_0|m_1 + n))-$*submodule of* $\mathbb{C}[M_{m_0|m_1+n,d}]$*, with highest weight*

$$(\lambda_1, \ldots, \lambda_{m_0}; \widetilde{\lambda}_1 - m_0, \widetilde{\lambda}_2 - m_0, \ldots).$$

3.8.2 The classical Schur $gl(n)-$modules

Given λ such that $\lambda_1 \leq n$, the *Schur module* $Schur_\lambda(n)$ is the subspace of $\mathbb{C}[M_{n,d}]$, $d \geq \lambda_1$, spanned by the set of all bitableaux $(X|D_\lambda^P)$ of shape λ and X is a Young tableau on the alphabet L.

Proposition 3.35. *The set*

$$\Big\{(X|D_\lambda^P); X \ \textit{standard}\ \Big\}$$

is a $\mathbb{C}-$*linear basis of* $Schur_\lambda(n)$.

Furthermore, $Schur_\lambda(n)$ *is an irreducible* $\mathbf{U}(gl(n))-$*submodule of* $\mathbb{C}[M_{n,d}]$*, with highest weight* $\widetilde{\lambda}$.

Let

$$D_\lambda = \left(\begin{array}{l} x_1 \ldots\ldots\ldots \quad x_{\lambda_1} \\ x_1 \ldots\ldots x_{\lambda_2} \\ \ldots\ldots \\ x_1 \ldots x_{\lambda_p} \end{array} \right)$$

denote the (proper) Deruyts tableau on the alphabet $L = \{x_1, x_2, \ldots, x_n\}$. The element

$$v_{\widetilde{\lambda}} = (D_\lambda|D_\lambda^P)$$

is the "canonical" highest weight vector of the irreducible $gl(n)-$module $Schur_\lambda(n)$, with highest weight $\widetilde{\lambda}$.

3.8.3 The classical Schur modules as $gl(n)-$submodules of Schur supermodules

Let λ be a partition such that $\lambda_1 \leq n$.

Given $m_0 \geq l(\lambda)$, $m_1, d \geq \lambda_1$, consider the Schur supermodule

$$Schur_\lambda(m_0, m_1 + n)$$

(clearly, $\lambda \in H(m_0, m_1 + n)$.)

The Schur module $Schur_\lambda(n)$ can be regarded as a $\mathbf{U}(gl(n))-$submodule of the $\mathbf{U}(gl(m_0|m_1+n))-$supermodule $Schur_\lambda(m_0, m_1+n)$.

Let $\mathfrak{p}$ be the Capelli epimorphism (Theorem 3.11)

$$\mathfrak{p} : Virt(m_0+m_1, n) \twoheadrightarrow \mathbf{U}(gl(n)), \qquad Ker(\mathfrak{p}) = \mathbf{Irr}.$$

Proposition 3.36. *The Schur module $Schur_\lambda(n)$ is invariant (as a subspace of $Schur_\lambda(m_0, m_1+n)$) with respect to the action of the subalgebra*

$$Virt(m_0+m_1, n) \subset \mathbf{U}(gl(m_0|m_1+n)).$$

Furthermore, for every element $\rho \in Virt(m_0+m_1, n)$, the action of ρ on the Schur module $Schur_\lambda(n)$ is the same of the action of its image $\mathfrak{p}(\rho)) \in \mathbf{U}(gl(n))$.

Proposition 3.37. ***(Vanishing Lemmas)*** *Let $v_{\widetilde{\mu}} = (D_\mu|D_\mu^P)$ be the "canonical" highest weight vector of the irreducible $gl(n)-$module $Schur_\mu(n)$, and let $\trianglelefteq$ denote the (partial) dominance order on partitions. We have*

$$\begin{aligned}
&\textit{If } |\mu| < |\lambda|, \textit{ then} && D^*_\lambda S(v_{\widetilde{\mu}}) = 0, \quad \forall S && (22)\\
&\textit{If } |\mu| < |\lambda|, \textit{ then} && C^*_\lambda S(v_{\widetilde{\mu}}) = 0, \quad \forall S && (23)\\
&\textit{If } |\mu| = |\lambda|, \ \mu \ntrianglerighteq \lambda, \textit{ then} && C^*_\lambda S(v_{\widetilde{\mu}}) = 0, \quad \forall S && (24)\\
&\textit{If } |\mu| = |\lambda|, \ \widetilde{\mu} \ntrianglerighteq \lambda, \textit{ then} && D^*_{\widetilde{\lambda}} S(v_{\widetilde{\mu}}) = 0, \quad \forall S && (25)\\
&\textit{If } |\mu| = |\lambda|, \ \mu \neq \lambda, \textit{ then} && D^*_\lambda C^*_\lambda \ C^*_\lambda S(v_{\widetilde{\mu}}) = 0, \quad \forall S && (26)\\
&\textit{If } |\mu| = |\lambda|, \ \mu \neq \lambda, \textit{ then} && C^*_\lambda D^*_\lambda \ D^*_\lambda S(v_{\widetilde{\mu}}) = 0, \quad \forall S && (27)\\
&\textit{If } \lambda \nsubseteq \mu, \textit{ then} && D^*_\lambda C^*_\lambda \ C^*_\lambda S(v_{\widetilde{\mu}}) = 0, \quad \forall S && (28)
\end{aligned}$$

Proof. The assertions of eqs. (22), (23), (24), (25), (26), (27) are special cases of standard elementary facts of the method of virtual variables (see e. g. [11]). About assertion of eq. (28), assume that $|\mu| \geq |\lambda|$ to avoid trivial cases (by eq. (23)). The action $C^*_\lambda S(v_{\widetilde{\mu}}) = C^*_\lambda S\big((D_\mu|D_\mu^P)\big)$ produces a linear combination of bitableaux $(T|D_\mu^P) \in Schur_\mu(m_0, m_1+n)$, where each tableau T contains exactly λ_i occurrences of the positive virtual symbols $\alpha_i \in A_0$. By *straightening* each of them (Theorem 3.30), the element $C^*_\lambda S\big((D_\mu|D_\mu^P)\big)$ is uniquely expressed as a linear combination of (super)standard tableaux

$$C^*_\lambda S\big((D_\mu|D_\mu^P)\big) = \sum_i \ (S_i|D_\mu^P) \in Schur_\mu(m_0, m_1+n), \tag{29}$$

where in each S_i the positive virtual symbols $\alpha_i \in A_0$ occupies a subshape $\lambda' \subseteq \mu$ such that $\lambda' \trianglerighteq \lambda$. If $\lambda \nsubseteq \mu$, any element $(S_i|D_\mu^P)$ in the canonical form (29) is such that $\lambda' \trianglerighteq \lambda$, $\lambda' \neq \lambda$. Then $D^*_\lambda C^*_\lambda\big((S_i|D_\mu^P)\big) = 0$, by skew-symmetry, and the assertion follows. ∎

We recall that, given a shape λ, the *hook length* $H(x)$ of a box x in the Ferrers diagram F_λ of the shape λ is the number of boxes that are in the same row to the right of it plus those boxes in the same column below it, plus one (for the box itself). The *hook number* the shape λ is the product $H(\lambda) = \prod_{x \in F_\lambda} \ H(x)$.

Proposition 3.38. ***(Regonati's Hook Lemma, [61])*** *Let $H(\lambda)$ denotes the hook number of the shape (partition) $\lambda \vdash k$. We have*

$$C^*_\lambda D_\lambda(v_{\tilde{\lambda}}) = \quad C^*_\lambda D_\lambda\big((D_\lambda|D^P_\lambda)\big) = (-1)^{\binom{k}{2}} \ H(\lambda) \cdot (C^*_\lambda|D^P_\lambda) \ (\lambda!)^{-1} \tag{30}$$

$$C^*_\lambda D^*_\lambda\big((D^*_\lambda|D^P_\lambda)\big) = (-1)^{\binom{k}{2}} \ H(\lambda) \cdot (C^*_\lambda|D^P_\lambda) \ (\lambda!)^{-1}. \tag{31}$$

Furthermore

$$D^*_\lambda C^*_\lambda\big((C^*_\lambda|D^P_\lambda) \ (\lambda!)^{-1}\big) = \quad (D^*_\lambda|D^P_\lambda) \tag{32}$$

4 The center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$

In order to make the notation lighter, in this section we simply write $1, 2, \ldots, n$ in place of $x_1, x_2, \ldots, x_n$.

Remark 4.1. *Throughout this section the role of Proposition 3.8 is ubiquitous: in order to prove that an element ρ is central in $\mathbf{U}(gl(n))$, we can simply prove that a virtual presentation ρ' in $Virt(m_0 + m_1, n)$ of ρ (to wit, an element ρ' that is mapped to ρ by the Capelli devirtualition epimorphism $\mathfrak{p}$, see Subsection 3.5.1) is an invariant for the adjoint action of $\mathbf{U}(gl(n))$ on $Virt(m_0 + m_1, n)$. In symbols*

$$ad(e_{ij})\,(\rho') = 0, \quad \forall e_{ij} \in gl(n).$$

4.1 The virtual form of the determinantal Capelli generators $\mathbf{H}_k(n)$ of 1893

4.1.1 The classical and the virtual definitions of the Capelli generators $\mathbf{H}_k(n)$, and main results

In the enveloping algebra $\mathbf{U}(gl(n))$, given any integer $k = 1, 2, \ldots, n$, consider the element (compare with Example 3.15)

$$\mathbf{H}_k(n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k] = \tag{33}$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathfrak{p}\big(e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}\big) = \tag{34}$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{cdet} \begin{pmatrix} e_{i_1,i_1} + (k-1) & e_{i_1,i_2} & \ldots & e_{i_1,i_k} \\ e_{i_2,i_1} & e_{i_2,i_2} + (k-2) & \ldots & e_{i_2,i_k} \\ \vdots & \vdots & \vdots & \\ e_{i_k,i_1} & e_{i_k,i_2} & \ldots & e_{i_k,i_k} \end{pmatrix} \tag{35}$$

where $\alpha \in A_0$ denotes *any* positive virtual symbol.

Remark 4.2. *Clearly, as apparent from the virtual presentation (34), each summand $[i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k]$ is skew-symmetric both in the left and the right* *sequences. As already observed in the Introduction, the "nonvirtual form" of the elements $\mathbf{H}_k(n)$ is harder to manage than their virtual form (see for example the proof of the centrality, Proposition 4.3 and Theorem 4.4).*

Proposition 4.3. *Since the adjoint representation acts by derivation, we have*

$$ad(e_{ij})\Big(\sum_{1 \leq i_1 < \cdots < i_k \leq n} e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k} \Big) = 0, \quad \forall e_{ij} \in gl(n).$$

Proof. In order to make the notation lighter, in this proof we write

$$\{ i_k \cdots i_2 \; i_1 | i_1 \; i_2 \cdots i_k \}$$

in place of $e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}$, that is

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k} =$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \{ i_k \cdots i_2 \; i_1 | i_1 \; i_2 \cdots i_k \}.$$

In the following, given an decreasing word $v = i_s \cdots i_2 i_1$ on the set $1, 2, \ldots, n$, we denote by $\overline{v} = i_1 i_2 \cdots i_s$ its reverse.

Without loss of generality, let us consider the adjoint action of an element e_{ij}, with $i > j$. note that the action on a summand $\{ i_k \cdots i_2 \; i_1 | i_1 \; i_2 \cdots i_k \}$ cannot be simultaneously nontrivial both "on the left" and "on the right".

The action "on the left" is nontrivial only on the summands of the form $\{ u \; \widehat{i} \; v \; j \; w | \overline{w} \; j \; \overline{v} \; \widehat{i} \; \overline{u} \}$ and it produces the element

$$\{ u \; \widehat{i} \; v \; i \; w | \overline{w} \; j \; \overline{v} \; \widehat{i} \; \overline{u} \} = (-1)^{l(v)} \{ u \; i \; v \; \widehat{j} \; w | \overline{w} \; j \; \overline{v} \; \widehat{i} \; \overline{u} \}$$

.

Consider the unique summand $\{ u \; i \; v \; \widehat{j} \; w | \overline{w} \; \widehat{j} \; \overline{v} \; i \; \overline{u} \}$; the adjoint action "on the right" produces the element

$$-\{ u \; i \; v \; \widehat{j} \; w | \overline{w} \; \widehat{i} \; \overline{v} \; j \; \overline{u} \} = -(-1)^{l(v)} \{ u \; i \; v \; \widehat{j} \; w | \overline{w} \; j \; \overline{v} \; \widehat{i} \; \overline{u} \}.$$

Then, after the action of the adjoint representation, the summands delete pairwise.

The action "on the right" is nontrivial only on the summands of the form $\{ u \; i \; v \; \widehat{j} \; w | \overline{w} \; \widehat{j} \; \overline{v} \; i \; \overline{u} \}$ and the reasoning is the same as before. $\square$

From Remark 4.1, it follows

Theorem 4.4. *The elements $\mathbf{H}_k(n)$ are central in $\mathbf{U}(gl(n))$.*

Let $\boldsymbol{\zeta}(n)^{(m)}$ denote the $m-$th filtration element of $\boldsymbol{\zeta}(n)$ with respect to the filtration induced by the standard filtration of $\mathbf{U}(gl(n))$.

Clearly,

$$\mathbf{H}_k(n) \in \boldsymbol{\zeta}(n)^{(m)},$$

for every $m \geq k$.

We recall the following fundamental result, indeed proved by Capelli in two papers ([23], [24]) with deceiving titles (for a faithful description of Capelli's original proof , quite simplified by means of the superalgebraic method of virtual variables, see [17], 1993)[5]

Theorem 4.5. *(Capelli, 1893)*

The set

$$\mathbf{H}_1(n), \mathbf{H}_2(n), \ldots, \mathbf{H}_n(n)$$

is a set of algebraically independent generators of the center $\boldsymbol{\zeta}(n)$ *of* $\mathbf{U}(gl(n))$.

We recall that $v_{\widetilde{\mu}} = (D_\mu | D^P_\mu)$ denoted the "canonical" highest weight vector of the Schur module $Schur_\mu(n)$, $\mu_1 \leq n$, which is indeed of weight $\widetilde{\mu}$ (Subsection 3.8.2).

Furthermore, we will write $\mathbf{H}_k(n)(v_{\widetilde{\mu}})$ to mean the action of the central element $\mathbf{H}_k(n)$ on $v_{\widetilde{\mu}}$.

Let $e^*_k(\widetilde{\mu}) \in \mathbb{N}$ denote the sum of the $k-$th "partial hooks" numbers of the shape μ, that is

$$e^*_k(\widetilde{\mu}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (\widetilde{\mu}_{i_1} + k - 1)(\widetilde{\mu}_{i_2} + k - 2) \cdots (\widetilde{\mu}_{i_k}). \tag{36}$$

The classical presentation (35) of the $\mathbf{H}_k(n)$'s implies the following result.

Proposition 4.6. *We have*

1. $\mathbf{H}_k(n)(v_{\widetilde{\mu}}) = e^*_k(\widetilde{\mu}) \cdot v_{\widetilde{\mu}}, \quad e^*_k(\widetilde{\mu}) \in \mathbb{Z}.$
2. $\mathbf{H}_k(n)(v_{\widetilde{\mu}}) = 0,$ *if* $\mu_1 < k$.

The "*virtual presentation*" (34) of the $\mathbf{H}_k(n)$'s leads to a further combinatorial description of the integer eigenvalues $e^*_k(\widetilde{\mu})$, which will turn out to be crucial in the section on *duality.*

Proposition 4.7. *We have*

$$e^*_k(\widetilde{\mu}) = \sum \ hstrip_\mu(k)!,$$

where the sum is extended to all "horizontal strips" [6] *of length* k *in the Ferrers diagram of the partition* μ*, and the symbol* $hstrip_\mu(k)!$ *denotes the products of the factorials of the cardinality of each horizontal component of the horizontal strip.*

[5] We are indebted to Kostant and Sahi ([44], p. 72), who wrote in 1991: "It is remarkable that, in some sense, Capelli was already aware of this connection! [10], p. 77".

[6] In this work, we use the expression *horizontal strip* in a generalized sense. To wit, a horizontal strip in a Ferrers diagram is a subset of cells such that no two cells in the subset appear in the same column.

Example 4.8. Let $\mu = (3,2)$. Given a bitableau $(S|D_\mu^P)$ in the supermodule $Schur_\mu(m_0|m_1+3)$, we will write $(S|$ in place of $(S|D_\mu^P)$, in order to simplify the notation.

In particular, we write

$$\left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right|$$

in place of the $gl(3)-$highest weight vector of the $gl(3)-$irreducible submodule $Schur_\mu(3)$ of $Schur_\mu(m_0|m_1+3)$:

$$(D_{(3,2)}|D^P_{(3,2)}) = \left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right|\left.\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right),$$

of weight $\widetilde{\mu} = (2,2,1)$.

The action of $\mathbf{H}_2(3)$ on $(D_\mu|D_\mu^P)$ is the same as the action of

$$e_{2,\alpha}e_{1,\alpha}e_{\alpha,1}e_{\alpha,2} + e_{3,\alpha}e_{1,\alpha}e_{\alpha,1}e_{\alpha,3} + e_{3,\alpha}e_{2,\alpha}e_{\alpha,2}e_{\alpha,3}, \quad \alpha \in A_0,\ |\alpha| = 0;$$

hence, we have to compute

$$\Big(D_{2,\alpha}D_{1,\alpha}D_{\alpha,1}D_{\alpha,2} + D_{3,\alpha}D_{1,\alpha}D_{\alpha,1}D_{\alpha,3} + D_{3,\alpha}D_{2,\alpha}D_{\alpha,2}D_{\alpha,3}\Big)\left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right|.$$

By considering the action of the "virtualizing part" of each summand, we have

$$\begin{aligned}
D_{\alpha,1}D_{\alpha,2}\left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right| &= -\left(\begin{array}{l}\alpha\ \alpha\ 3\\ 1\ 2\end{array}\right| + \left(\begin{array}{l}\alpha\ 2\ 3\\ 1\ \alpha\end{array}\right| - \left(\begin{array}{l}1\ \alpha\ 3\\ \alpha\ 2\end{array}\right| - \left(\begin{array}{l}1\ 2\ 3\\ \alpha\ \alpha\end{array}\right|,\\
D_{\alpha,1}D_{\alpha,3}\left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right| &= \left(\begin{array}{l}\alpha\ 2\ \alpha\\ 1\ 2\end{array}\right| + \left(\begin{array}{l}1\ 2\ \alpha\\ \alpha\ 2\end{array}\right|,\\
D_{\alpha,2}D_{\alpha,3}\left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right| &= -\left(\begin{array}{l}1\ \alpha\ \alpha\\ 1\ 2\end{array}\right| - \left(\begin{array}{l}1\ 2\ \alpha\\ 1\ \alpha\end{array}\right|.
\end{aligned}$$

Notice that the two occurrences of α distribute in all horizontal strips of length 2 in the Ferrers diagram of the partition $\mu = (3,2)$.

By considering the action of the "devirtualizing part" of each summand, we have

$$\begin{aligned}
&D_{2,\alpha}D_{1,\alpha}\Big(-\left(\begin{array}{l}\alpha\ \alpha\ 3\\ 1\ 2\end{array}\right| + \left(\begin{array}{l}\alpha\ 2\ 3\\ 1\ \alpha\end{array}\right| - \left(\begin{array}{l}1\ \alpha\ 3\\ \alpha\ 2\end{array}\right| - \left(\begin{array}{l}1\ 2\ 3\\ \alpha\ \alpha\end{array}\right|\Big) =\\
&= \left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right| - \left(\begin{array}{l}2\ 1\ 3\\ 1\ 2\end{array}\right| + \left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right| + \left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right| + \left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right| - \left(\begin{array}{l}1\ 2\ 3\\ 2\ 1\end{array}\right| =\\
&\qquad = 6\left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right|,
\end{aligned}$$

$$D_{3,\alpha}D_{1,\alpha}\Big(\left(\begin{array}{l}\alpha\ 2\ \alpha\\ 1\ 2\end{array}\right| + \left(\begin{array}{l}1\ 2\ \alpha\\ \alpha\ 2\end{array}\right|\Big) = 3\left(\begin{array}{l}1\ 2\ 3\\ 1\ 2\end{array}\right|,$$

$$D_{3,\alpha}D_{2,\alpha}\Big(-\left(\begin{array}{l} 1\ \alpha\ \alpha \\ 1\ 2 \end{array} \right| - \left(\begin{array}{l} 1\ 2\ \alpha \\ 1\ \alpha \end{array} \right| \Big) = 3\ \left(\begin{array}{l} 1\ 2\ 3 \\ 1\ 2 \end{array} \right|.$$

Therefore,

$$\Big(D_{2,\alpha}D_{1,\alpha}D_{\alpha,1}D_{\alpha,2}+D_{3,\alpha}D_{1,\alpha}D_{\alpha,1}D_{\alpha,3}+D_{3,\alpha}D_{2,\alpha}D_{\alpha,2}D_{\alpha,3}\Big)\ \left(\begin{array}{l} 1\ 2\ 3 \\ 1\ 2 \end{array} \right| = \\ = 12\ \left(\begin{array}{l} 1\ 2\ 3 \\ 1\ 2 \end{array} \right|.$$

Notice that, since $\widetilde{\mu} = (2, 2, 1)$, according to Corollary 4.6 and eq. (36), we have

$$e_2^*((2,2,1)) = \sum_{1 \leq i_1 < i_2 \leq 3} (\widetilde{\mu}_{i_1} + 2 - 1)(\widetilde{\mu}_{i_2}) = \\ = (2+2-1)2 + (2+2-1)1 + (2+2-1)1 = 12.$$

□

Proof of Proposition 4.7. The action of each summand of the "virtualizing part"

$$e_{\alpha,i_1}e_{\alpha,i_2}\cdots e_{\alpha,i_k}$$

distributes the k occurrences of α in all horizontal strips of length k (with column positions $i_1, i_2, \ldots, i_k$) in the Ferrers diagram of the partition μ, with signs - according to Proposition 3.27 - since $|e_{\alpha,i_h}| = 1$. By applying the "devirtualizing part"

$$e_{i_k,\alpha}\cdots e_{i_2,\alpha}e_{i_1,\alpha}$$

it is easy to see that, for each horizontal strip, we obtain a sum of tableaux in which:

- by skew-symmetry, in any horizontal component, the occurrences of α are replaced exactly by all the permutations of the elements that have been previously polarized;
- the sign that is produced at this stage is the same as the sign produced by the action of the "virtualizing part" times the product of the signs of the permutations of the elements in each horizontal component.

By reordering each horizontal component - again by skew-symmetry - all the signs cancel out, and therefore, it just appear an integer coefficient that is the product of the factorials of the lengths of the horizontal components. □

As usual in the theory of symmetric functions, given a shape

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_p),\ \lambda_1 \leq n,$$

we set

$$\mathbf{H}_\lambda(n) = \mathbf{H}_{\lambda_1}(n)\mathbf{H}_{\lambda_2}(n)\cdots\mathbf{H}_{\lambda_p}(n).$$

By convention, if λ is the empty partition, we set $\mathbf{H}_\emptyset(n) = \mathbf{1} \in \boldsymbol{\zeta}(n)$.

From Theorem 4.5, one infers

Corollary 4.9. *The set*

$$\left\{\mathbf{H}_\lambda(n); \lambda_1 \leq n,\ |\lambda| \leq m\ \right\}$$

is a linear basis of $\boldsymbol{\zeta}(n)^{(m)}$.

4.1.2 The Koszul isomorphism and the Laplace expansion into column bitableaux

In this subsection, we will consider the Capelli determinantal generators $\mathbf{H}_k(n)$ from the point of view of the *Bitableaux correspondence and Koszul map Theorems* ([17], Thms. 1 and 2, see also [12], [46]). We will show that the Capelli determinants in $\mathbf{U}(gl(n))$ expand into "*column bitableaux*" in the same way as the determinants of matrices with commutative entries expand into ordinary monomials, once the integers on the main diagonal are omitted. Furthermore, column bitableaux are indeed the analogues - in $\mathbf{U}(gl(n))$ - of monomials in a polynomial algebra, since they are invariant with respect to permutations of their rows, as monomials are invariant with respect to permutations of their factors.

Proposition 4.10. *Given a $k-$tuple $(i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, the element*

$$[i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k] = (-1)^{\binom{k}{2}} [i_1 i_2 \cdots i_k | i_1 i_2 \cdots i_k] =$$

$$= \mathbf{cdet}\begin{pmatrix} e_{i_1,i_1} + (k-1) & e_{i_1,i_2} & \ldots & e_{i_1,i_k} \\ e_{i_2,i_1} & e_{i_2,i_2} + (k-2) & \ldots & e_{i_2,i_k} \\ \vdots & \vdots & \vdots & \\ e_{i_k,i_1} & e_{i_k,i_2} & \ldots & e_{i_k,i_k} \end{pmatrix} \in \mathbf{U}(gl(n))$$

expand into column bitableaux as

$$(-1)^{\binom{k}{2}} \sum_\sigma \ (-1)^{|\sigma|} \left[\begin{array}{c|c} i_1 & i_{\sigma(1)} \\ i_2 & i_{\sigma(2)} \\ \vdots & \vdots \\ i_k & i_{\sigma(k)} \end{array}\right], \qquad (37)$$

where the summands

$$\left[\begin{array}{c|c} i_1 & i_{\sigma(1)} \\ i_2 & i_{\sigma(2)} \\ \vdots & \vdots \\ i_k & i_{\sigma(k)} \end{array}\right]$$

are the column bitableaux

$$\mathfrak{p}\big(e_{i_1,\gamma_1}e_{i_2,\gamma_2}\cdots e_{i_k,\gamma_k}e_{\gamma_1,i_{\sigma(1)}}e_{\gamma_2,i_{\sigma(2)}}\cdots e_{\gamma_k,i_{\sigma(k)}}\big),$$

the symbols $\gamma_1, \gamma_2, \cdots, \gamma_k$ are distinct positive virtual symbols in A_0, and the summation is extended to all permutations σ of the set $\{1, 2, \ldots, k\}$.

In plain words, the expression of $[i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k]$ in terms of column bitableaux in (37) formally eliminates the "*queues*" due to the integer summands that appear on the main diagonal. Furthermore, this fact leads to a *third* combinatorial description of the eigenvalues of the central Capelli generators on irreducible representations; this phenomenon implies, in turn, a noteworthy generating function formula for *permutation statistics*.

Example 4.11. The Capelli element $\mathbf{H}_2(2) = [21|12]$ equals

$$\mathbf{cdet}\begin{pmatrix} e_{1,1}+1 & e_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix} = [21|12] = -[12|12] = -\left[\begin{array}{c|c} 1 & 1 \\ 2 & 2 \end{array}\right] + \left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \end{array}\right],$$

where

$$-\left[\begin{array}{c|c} 1 & 1 \\ 2 & 2 \end{array}\right] = e_{1,1}e_{2,2} = -\left[\begin{array}{c|c} 2 & 2 \\ 1 & 1 \end{array}\right] = e_{2,2}e_{1,1},$$

and

$$\left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \end{array}\right] = -e_{1,2}e_{2,1} + e_{1,1} = \left[\begin{array}{c|c} 2 & 1 \\ 1 & 2 \end{array}\right] = -e_{2,1}e_{1,2} + e_{2,2}.$$

□

The actions (on highest weight vectors) of the vertical bitableaux that appear in Proposition 4.10 admit a remarkable combinatorial description.

Given a shape μ, $\mu_1 \leq n$, a subset $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and a permutation σ of $\{i_1, i_2, \ldots, i_k\}$, consider the integer

$$\Gamma_\sigma(\mu; i_1, i_2, \ldots, i_k) = (\widetilde{\mu}_{i_1})^{h_\sigma(i_1)}(\widetilde{\mu}_{i_2})^{h_\sigma(i_2)}\cdots(\widetilde{\mu}_{i_k})^{h_\sigma(i_k)},$$

where

$$h_\sigma(j) = 1, \qquad j \in \underline{n}$$

if $j \in \underline{n}$ is a maximum element in a cycle of the cycle decomposition of the permutation σ of the subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, and 0 otherwise.

Example 4.12. Let $n \geq 12$, $k = 8$, $(i_1, i_2, \ldots, i_8) = (2, 4, 5, 6, 7, 9, 11, 12)$. Consider the permutation $\sigma = (6\ 2\ 4)(9\ 5)(11\ 7)(12)$. Let μ be a shape, $\mu_1 \leq n$. Then

$$\Gamma_\sigma(\mu; 2, 4, 5, 6, 7, 9, 11, 12) = \widetilde{\mu}_6\widetilde{\mu}_9\widetilde{\mu}_{11}\widetilde{\mu}_{12}.$$

□

Proposition 4.13. *The action of the vertical bitabeau*

$$(-1)^{\binom{k}{2}}(-1)^{|\sigma|}\left[\begin{array}{c|c} i_1 & i_{\sigma(1)} \\ i_2 & i_{\sigma(2)} \\ \vdots & \vdots \\ i_k & i_{\sigma(k)} \end{array}\right]$$

on the highest weight vector $v_{\widetilde{\mu}}$ of weight $(\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots, \widetilde{\mu}_n)$ equals

$$\Gamma_\sigma(\mu; i_1, i_2, \ldots, i_k) \cdot v_{\widetilde{\mu}}.$$

Proof. Recall that

$$\begin{aligned}
&(-1)^{\binom{k}{2}}(-1)^{|\sigma|}\left[\begin{array}{c|c} i_1 & i_{\sigma(1)} \\ i_2 & i_{\sigma(2)} \\ \vdots & \vdots \\ i_k & i_{\sigma(k)} \end{array}\right] = \\
&= (-1)^{\binom{k}{2}}(-1)^{|\sigma|}\mathfrak{p}\big(e_{i_1,\gamma_1}\ e_{i_2,\gamma_2} \cdots e_{i_k,\gamma_k} \cdot e_{\gamma_1,i_{\sigma(1)}}\ e_{\gamma_2,i_{\sigma(2)}} \cdots e_{\gamma_k,i_{\sigma(k)}}\big) \\
&= (-1)^{|\sigma|}\mathfrak{p}\big(e_{i_k,\gamma_k} \cdots e_{i_2,\gamma_2}\ e_{i_1,\gamma_1} \cdot e_{\gamma_1,i_{\sigma(1)}}\ e_{\gamma_2,i_{\sigma(2)}} \cdots e_{\gamma_k,i_{\sigma(k)}}\big).
\end{aligned}$$

Given a shape μ, $\mu_1 \leq n$, let $v_{\widetilde{\mu}} = (D_\mu | D_\mu^P)$ the canonical highest weight vector of the Schur module $Schur_\mu(n)$.

We have to study the action of

$$(-1)^{|\sigma|} e_{i_k,\gamma_k} \cdots e_{i_2,\gamma_2}\ e_{i_1,\gamma_1} \cdot e_{\gamma_1,i_{\sigma(1)}}\ e_{\gamma_2,i_{\sigma(2)}} \cdots e_{\gamma_k,i_{\sigma(k)}}$$

on the bitableau $(D_\mu | D_\mu^P) \in \mathbb{C}[M_{n,d}]$.

When the "virtualizing part" $e_{\gamma_1,i_{\sigma(1)}}\ e_{\gamma_2,i_{\sigma(2)}} \cdots e_{\gamma_k,i_{\sigma(k)}}$ acts, it distributes the virtual symbols $\gamma_1, \gamma_2, \ldots, \gamma_k$ into the columns $i_1, i_2, \ldots, i_k$ of the tableau D_μ in all possible ways with some signs. These signs will be canceled by the the action of the "devirtualizing part" $e_{i_k,\gamma_k} \cdots e_{i_2,\gamma_2}\ e_{i_1,\gamma_1}$.

Furthermore, by skew-symmetry, the configurations of the virtual symbols that remain non zero after the action of the "devirtualizing part" are those in that virtual symbols associated to elements in $\{i_1, i_2, \ldots, i_k\}$ that belong to the same cycle of the permutation σ appear in the same row of the tableau D_μ. Finally, by reordering the rows, the global sign $(-1)^{|\sigma|}$ of the permutation σ cancels in each summand.

Therefore, the eigenvalue is nothing but that the number of ways of distributing the virtual symbols so that virtual symbols associated to elements that belong to the same cycle appear in the same row, that is $\Gamma_\sigma(\mu; i_1, i_2, \ldots, i_k)$. $\square$

Example 4.14. Let $n \geq 9$ and $k = 7$, $(i_1, i_2, \ldots, i_7) = (1, 2, 3, 4, 5, 7, 9)$, and consider the column bitableau

$$(-1)^{\binom{7}{2}} \left[\begin{array}{c|c} 1 & \sigma(1) \\ 2 & \sigma(2) \\ 3 & \sigma(3) \\ 4 & \sigma(4) \\ 5 & \sigma(5) \\ 7 & \sigma(7) \\ 9 & \sigma(9) \end{array} \right] = - \left[\begin{array}{c|c} 1 & 5 \\ 2 & 1 \\ 3 & 9 \\ 4 & 7 \\ 5 & 2 \\ 7 & 4 \\ 9 & 3 \end{array} \right]. \tag{38}$$

Since the permutation σ of the set $\{1, 2, 3, 4, 5, 7, 9\}$ is even and has cycle decomposition

$$\sigma = (521)(74)(93),$$

then the column bitableau (38) acts on the highest weight vectors $v_{\widetilde{\mu}}$ (of highest weight $\widetilde{\mu}$) just multiplying it by the integer

$$\Gamma_\sigma(\mu; 1, 2, 3, 4, 5, 7, 9) = \widetilde{\mu}_5 \ \widetilde{\mu}_7 \ \widetilde{\mu}_9.$$

□

By combining Propositions 4.10 and 4.13, one infers a third combinatorial description of the eigenvalues of the Capelli generators $\mathbf{H}_k(n)$ on the irreducible modules $Schur_\mu(n)$ of highest weight $\widetilde{\mu}$.

Proposition 4.15. *The eigenvalue of the action of the central generator $\mathbf{H}_k(n)$ on the irreducible module $Schur_\mu(n)$ equals*

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \ \sum_\sigma \ \Gamma_\sigma(\mu; i_1, i_2, \ldots, i_k),$$

where the inner sum ranges over all permutations σ of the set $\{i_1, i_2, \ldots, i_k\}$.

By comparing Propositions 4.6, 4.7, 4.15, it follows

Corollary 4.16. *Let μ be a shape, $\mu_1 \leq n$, $k \leq n$. Then*

$$\begin{aligned} e^*_k(\widetilde{\mu}) &= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (\widetilde{\mu}_{i_1} + k - 1)(\widetilde{\mu}_{i_2} + k - 2) \cdots (\widetilde{\mu}_{i_k}) \\ &= \sum hstrip_\mu(k)! \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \ \sum_\sigma \ \Gamma_\sigma(\mu; i_1, i_2, \ldots, i_k). \end{aligned}$$

Passing to polynomials, Corollary 4.16 implies (in the special case $k = n$) a generating function formula for *permutation statistics*, which is in turn equivalent - via the *Foata correspondence* - to a result of Wilf [72].

Keeping the notation for the coefficients $\Gamma_\sigma(\mu; i_1, i_2, \ldots, i_k)$, consider the generating polynomial

$$\mathbf{W}_n = \sum_\sigma \ x_1^{h_\sigma(1)} x_2^{h_\sigma(2)} \cdots x_n^{h_\sigma(n)},$$

where σ ranges over all permutations of the set $\{1, 2, \ldots, n\}$.

Proposition 4.17. *We have*

$$\mathbf{W}_n = \sum_{\sigma} \; x_1^{h_\sigma(1)} x_2^{h_\sigma(2)} \cdots x_n^{h_\sigma(n)} = e_n^*(x_1, x_2, \ldots, x_n),$$

where

$$e_n^*(x_1, x_2, \ldots, x_n) = (x_1 + n - 1)(x_2 + n - 2) \cdots x_n,$$

the $n-$th elementary shifted symmetric polynomial in n variables.

Example 4.18. Let $n = 3$. Then

$$\mathbf{W}_3 = x_1x_2x_3 + x_1x_3 + 2x_2x_3 + 2x_3,$$

that equals

$$e_3^*(x_1, x_2, x_3) = (x_1 + 2)(x_2 + 1)x_3.$$

$\square$

Remark 4.19. *The approach to Capelli elements by expansion into vertical bitableaux naturally extends to the other classes of central elements studied in the present work. For the sake of brevity, we leave this extension to the reader.*

4.2 The virtual form of the permanental Nazarov/Umeda elements $\mathbf{I}_k(n)$

In this section we provide the virtual form of the set of the preimages in $\boldsymbol{\zeta}(n)$ - with respect to the Harish-Chandra isomorphism - of the sequence of *shifted complete symmetric polynomials* $\mathbf{h}_k^*(x_1, x_2, \ldots, x_n)$, for every $k \in \mathbb{Z}^+$, see [56] and [49], Theorem 4.9.

The central elements $\mathbf{I}_k(n)$, $k \in \mathbb{Z}^+$, coincide (see [12]) with the "permanental generators" of $\boldsymbol{\zeta}(n)$ originally discovered and studied - through the machinery of *Yangians* - by Nazarov [52] and later described by Umeda [69] as sums of column permanents in $\mathbf{U}(gl(n))$ (see also [51], [53], and Turnbull [66]).

Definition 4.20. *For every $k \in \mathbb{Z}^+$, set*

$$\begin{aligned}\mathbf{I}_k(n) &= \sum_{(h_1,h_2,\ldots,h_n)} (h_1!h_2!\cdots h_n!)^{-1} \; [n^{h_n} \cdots 2^{h_2}1^{h_1}|1^{h_1}2^{h_2} \cdots n^{h_n}]^* = \\ &= \sum_{(h_1,h_2,\ldots,h_n)} (h_1!h_2!\cdots h_n!)^{-1} \; \mathfrak{p}\big(e_{n,\beta}^{h_n} \cdots e_{2,\beta}^{h_2} e_{1,\beta}^{h_1} e_{\beta,1}^{h_1} e_{\beta,2}^{h_2} \cdots e_{\beta,n}^{h_n}\big),\end{aligned}$$

where $\beta \in A_1$ denotes any negative virtual symbol, the sum is extended to all $n-$tuples $(h_1, h_2, \ldots, h_n)$ such that $h_1 + h_2 + \cdots + h_n = k$.

Remark 4.21. *Clearly, as apparent from the virtual presentation, each summand*

$$[n^{h_n} \cdots 2^{h_2}1^{h_1}|1^{h_1}2^{h_2} \cdots n^{h_n}]^*$$

is symmetric both in the left and the right sequences. In "nonvirtual form", the summands can be written as column permanent in the algebra $\mathbf{U}(gl(n))$ (see also Example 3.16). As already observed in the Introduction, the nonvirtual form of the elements $\mathbf{I}_k(n)$ is much harder to manage than their virtual form (see for example the proof of the centrality, Proposition 4.23 and Theorem 4.24).

Example 4.22.

$$\mathbf{I}_3(3) = \frac{1}{3!}[111|111]^* + \frac{1}{2!}[211|112]^* + \frac{1}{2!}[311|113]^* + \frac{1}{2!}[221|122]^* + [321|123]^* +$$
$$+ \frac{1}{2!}[331|133]^* + \frac{1}{3!}[222|222]^* + \frac{1}{2!}[322|223]^* + \frac{1}{2!}[332|233]^* + \frac{1}{3!}[333|333]^* =$$
$$= \frac{1}{3!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,1}-1 & e_{1,1} \\ e_{1,1}-2 & e_{1,1}-1 & e_{1,1} \\ e_{1,1}-2 & e_{1,1}-1 & e_{1,1} \end{pmatrix} + \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,1}-1 & e_{1,2} \\ e_{1,1}-2 & e_{1,1}-1 & e_{1,2} \\ e_{2,1} & e_{2,1} & e_{2,2} \end{pmatrix} +$$
$$+ \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,1}-1 & e_{1,3} \\ e_{1,1}-2 & e_{1,1}-1 & e_{1,3} \\ e_{3,1} & e_{3,1} & e_{3,3} \end{pmatrix} + \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,2} & e_{1,2} \\ e_{2,1} & e_{2,2}-1 & e_{2,2} \\ e_{2,1} & e_{2,2}-1 & e_{2,2} \end{pmatrix} +$$
$$+ \mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2}-1 & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix} + \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,3} & e_{1,3} \\ e_{3,1} & e_{3,3}-1 & e_{3,3} \\ e_{3,1} & e_{3,3}-1 & e_{3,3} \end{pmatrix} +$$
$$+ \frac{1}{3!}\mathbf{cper}\begin{pmatrix} e_{2,2}-2 & e_{2,2}-1 & e_{2,2} \\ e_{2,2}-2 & e_{2,2}-1 & e_{2,2} \\ e_{2,2}-2 & e_{2,2}-1 & e_{2,2} \end{pmatrix} + \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{2,2}-2 & e_{2,2}-1 & e_{2,3} \\ e_{2,2}-2 & e_{2,2}-1 & e_{2,3} \\ e_{3,2} & e_{3,2} & e_{3,3} \end{pmatrix} +$$
$$+ \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{2,2}-2 & e_{2,3} & e_{2,3} \\ e_{3,2} & e_{3,3}-1 & e_{3,3} \\ e_{3,2} & e_{3,3}-1 & e_{3,3} \end{pmatrix} + \frac{1}{3!}\mathbf{cper}\begin{pmatrix} e_{3,3}-2 & e_{3,3}-1 & e_{3,3} \\ e_{3,3}-2 & e_{3,3}-1 & e_{3,3} \\ e_{3,3}-2 & e_{3,3}-1 & e_{3,3} \end{pmatrix}.$$

□

Proposition 4.23. *Since the adjoint representation acts by derivation, we have*

$$ad(e_{ij})\Big(\sum_{(h_1,h_2,\ldots,h_n)} (h_1!h_2!\cdots h_n!)^{-1}\ e_{n,\beta}^{h_n}\cdots e_{2,\beta}^{h_2}e_{1,\beta}^{h_1}e_{\beta,1}^{h_1}e_{\beta,2}^{h_2}\cdots e_{\beta,n}^{h_n}\Big) = 0,$$

for every $e_{ij} \in gl(n)$.

Proof. In order to make the notation lighter, in this proof we write

$$\{n^{h_n}\cdots 2^{h_2}1^{h_1}|1^{h_1}2^{h_2}\cdots n^{h_n}\}^*$$

in place of $e_{n,\beta}^{h_n}\cdots e_{2,\beta}^{h_2}e_{1,\beta}^{h_1}e_{\beta,1}^{h_1}e_{\beta,2}^{h_2}\cdots e_{\beta,n}^{h_n}$, that is

$$\sum_{(h_1,h_2,\ldots,h_n)} (h_1!h_2!\cdots h_n!)^{-1}\ e_{n,\beta}^{h_n}\cdots e_{2,\beta}^{h_2}e_{1,\beta}^{h_1}e_{\beta,1}^{h_1}e_{\beta,2}^{h_2}\cdots e_{\beta,n}^{h_n} =$$
$$= \sum_{(h_1,h_2,\ldots,h_n)} (h_1!h_2!\cdots h_n!)^{-1}\ \{n^{h_n}\cdots 2^{h_2}1^{h_1}|1^{h_1}2^{h_2}\cdots n^{h_n}\}^*.$$

In the following, given an decreasing word $v = i_s \cdots i_2 i_1$ on the set $1, 2, \ldots, n$, we denote by $\overline{v} = i_1 i_2 \cdots i_s$ its reverse.

Without loss of generality, let us consider the adjoint action of the element e_{12}, that is $i = 1,\ j = 2$.

Since

$$\{n^{h_n} \cdots 2^{h_2} 1^{h_1} | 1^{h_1} 2^{h_2} \cdots n^{h_n}\}^* = \{1^{h_1} 2^{h_2} \cdots n^{h_n} | 1^{h_1} 2^{h_2} \cdots n^{h_n}\}^*,$$

we write

$$\sum_{(h_1, h_2, \ldots, h_n)} (h_1! h_2! \cdots h_n!)^{-1}\ \{n^{h_n} \cdots 2^{h_2} 1^{h_1} | 1^{h_1} 2^{h_2} \cdots n^{h_n}\}^*$$

in the form

$$\sum_w \sum_{h=0}^{k} \Big(\sum_{j=0}^{h} (j!)^{-1}\ (h - j!)^{-1}\ (w!)^{-1}\ \{1^j\ 2^{h-j}\ w | 1^j\ 2^{h-j}\ w\}^* \Big),$$

where the outer sum is extended over all words $w = 3^{h_3}\ 4^{h_4} \cdots n^{h_n}$, with with $h_3\ +\ h_4\ + \cdots +\ h_n = k - h$, and $w! = h_3!\ h_4!\ \cdots\ h_n!$.

We claim that the adjoint action of e_{12} on

$$\sum_{j=0}^{h} (j!)^{-1}\ (h - j!)^{-1} (w!)^{-1} \{1^j\ 2^{h-j}\ w | 1^j\ 2^{h-j}\ w\}^*$$

equals zero. Indeed it produces

$$\sum_{j=0}^{h-1} (j!)^{-1}\ ((h - j - 1)!)^{-1}\ (w!)^{-1}\ \{1^{j+1}\ 2^{h-j-1}\ w | 1^j\ 2^{h-j}\ w\}^* -$$

$$- \sum_{j=1}^{h}\ ((j-1)!)^{-1}\ ((h-j)!)^{-1}\ (w!)^{-1}\ (w!)^{-1} \{1^j\ 2^{h-j}\ w | 1^{j-1}\ 2^{h-j+1}\ w\}^*$$

By performing in the second sum the substitution of variables $j = p + 1$, we get

$$\sum_{j=0}^{h-1} (j!)^{-1}\ ((h - j - 1)!)^{-1}\ (w!)^{-1}\ \{1^{j+1}\ 2^{h-j-1}\ w | 1^j\ 2^{h-j}\ w\}^* -$$

$$- \sum_{p=0}^{h-1}\ (p!)^{-1}\ ((h - p - 1)!)^{-1}\ (w!)^{-1}\ \{1^{p+1}\ 2^{h-p-1}\ w | 1^p\ 2^{h-p}\ w\}^* = 0.$$

$\square$

From Remark 4.1, it follows

Theorem 4.24. *The elements $\mathbf{I}_k(n)$ are central in $\mathbf{U}(gl(n))$.*

Clearly,

$$\mathbf{I}_k(n) \in \boldsymbol{\zeta}(n)^{(m)},$$

for every $m \geq k$.

The proofs of the following results are almost trivial, as a consequence of the definition of the elements $\mathbf{I}_k(n)$ in terms of their *virtual presentations* (Definition 4.20).

Theorem 4.25. *We have:*

1. *We have*
$$\mathbf{I}_k(n)(v_{\widetilde{\mu}}) = h_k^*(\widetilde{\mu}) \cdot v_{\widetilde{\mu}}, \quad h_k^*(\widetilde{\mu}) \in \mathbb{N}$$
with
$$h_k^*(\widetilde{\mu}) = \sum \ vstrip_\mu(k)!,$$
where the sum is extended to all "vertical strips" [7] *of length k in the Ferrers diagram of the partition μ, and the symbol $vstrip_\mu(k)!$ denotes the products of the factorials of the cardinality of each vertical component of the vertical strip.*

2. *Therefore:*
$$\mathbf{I}_k(n)(v_{\widetilde{\mu}}) = h_k^*(\widetilde{\mu}) \cdot v_{\widetilde{\mu}}, \quad h_k^*(\widetilde{\mu}) \in \mathbb{Z},$$
where
$$h_k^*(\widetilde{\mu}) = \sum_{1 \leq i_1 \leq i_2 < \cdots \leq i_k \leq n} (\widetilde{\mu}_{i_1} - k + 1)(\widetilde{\mu}_{i_2} - k + 2) \cdots (\widetilde{\mu}_{i_k}) \tag{39}$$

3. *If $\widetilde{\mu}_1 < k$, then*
$$\mathbf{I}_k(n)(v_{\widetilde{\mu}}) = 0.$$

Example 4.26. Let $\lambda = (2,2,1)$. Given a bitableau $(S|D_\lambda^P)$ in the supermodule $Schur_\lambda(m_0|m_1+3)$, we will write $(S|$ in place of $(S|D_\lambda^P)$, in order to simplify the notation.

In particular, we write

$$\left(\begin{array}{c|} 1\ 2 \\ 1\ 2 \\ 1\ \ \ \end{array} \right.$$

in place of the $gl(3)-$highest weight vector of the $gl(3)-$irreducible submodule of $Schur_\lambda(3)$ of $Schur_\lambda(m_0|m_1+3)$:

$$(D_{(2,2,1)}|D_{(2,2,1)}^P) = \left(\begin{array}{c|c} 1\ 2 & 1\ 2 \\ 1\ 2 & 1\ 2 \\ 1\ \ \ & 1\ \ \ \end{array} \right),$$

[7] In this work, we use the expression *vertical strip* in a generalized sense. To wit, a vertical strip in a Ferrers diagram is a subset of cells such that no two cells in the subset appear in the same row.

of weight $\widetilde{\lambda} = (3, 2)$.

The action of $\mathbf{I}_2(3)$ on $(D_\lambda | D^P_{\widetilde{\lambda}})$ is the same as the action of

$$
\begin{aligned}
&(2!)^{-1}\ e_{1,\beta}e_{1,\beta}e_{\beta,1}e_{\beta,1} + (2!)^{-1}\ e_{2,\beta}e_{2,\beta}e_{\beta,2}e_{\beta,2} + (2!)^{-1}\ e_{3,\beta}e_{3,\beta}e_{\beta,3}e_{\beta,3} + \\
&+ e_{2,\beta}e_{1,\beta}e_{\beta,1}e_{\beta,2} + e_{3,\beta}e_{1,\beta}e_{\beta,1}e_{\beta,3} + e_{3,\beta}e_{2,\beta}e_{\beta,2}e_{\beta,3}, \quad \beta \in A_1,\ |\beta| = 1;
\end{aligned}
$$

hence, we have to compute

$$
\begin{aligned}
\Big((2!)^{-1}\ & D_{1,\beta}D_{1,\beta}D_{\beta,1}D_{\beta,1} + (2!)^{-1}\ D_{2,\beta}D_{2,\beta}D_{\beta,2}D_{\beta,2} + \\
& + (2!)^{-1}\ D_{3,\beta}D_{3,\beta}D_{\beta,3}D_{\beta,3} + + D_{2,\beta}D_{1,\beta}D_{\beta,1}D_{\beta,2} + \\
& + D_{3,\beta}D_{1,\beta}D_{\beta,1}D_{\beta,3} + D_{3,\beta}D_{2,\beta}D_{\beta,2}D_{\beta,3}\Big) \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array}\right| .
\end{aligned}
$$

By considering the action of the "virtualizing part" of each summand, we have

$$
\begin{aligned}
(2!)^{-1}\ D_{\beta,1}D_{\beta,1} \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array}\right| &= \left(\begin{array}{l} \beta\ 2 \\ \beta\ 2 \\ 1 \end{array}\right| + \left(\begin{array}{l} \beta\ 2 \\ 1\ 2 \\ \beta \end{array}\right| + \left(\begin{array}{l} 1\ 2 \\ \beta\ 2 \\ \beta \end{array}\right| , \\
(2!)^{-1}\ D_{\beta,2}D_{\beta,2} \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array}\right| &= \left(\begin{array}{l} 1\ \beta \\ 1\ \beta \\ 1 \end{array}\right| , \\
(2!)^{-1}\ D_{\beta,3}D_{\beta,3} \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array}\right| &= 0, \\
D_{\beta,1}D_{\beta,2} \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array}\right| &= \left(\begin{array}{l} \beta\ 2 \\ 1\ \beta \\ 1 \end{array}\right| + \left(\begin{array}{l} 1\ \beta \\ \beta\ 2 \\ 1 \end{array}\right| + \left(\begin{array}{l} 1\ \beta \\ 1\ 2 \\ \beta \end{array}\right| + \left(\begin{array}{l} 1\ 2 \\ 1\ \beta \\ \beta \end{array}\right| , \\
D_{\alpha,1}D_{\alpha,3} \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array}\right| &= 0, \\
D_{\alpha,2}D_{\alpha,3} \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array}\right| &= 0.
\end{aligned}
$$

Notice that the two occurrences of β distribute in all vertical strips of length 2 in the Ferrers diagram of the partition $\lambda = (2, 2, 1)$.

By considering the action of the "devirtualizing part" of each summand, we have

$$
D_{1,\beta}D_{1,\beta}\left(\left(\begin{array}{l} \beta\ 2 \\ \beta\ 2 \\ 1 \end{array}\right| + \left(\begin{array}{l} \beta\ 2 \\ 1\ 2 \\ \beta \end{array}\right| + \left(\begin{array}{l} 1\ 2 \\ \beta\ 2 \\ \beta \end{array}\right| \right) = 6 \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array}\right| ,
$$

$$D_{2,\beta}D_{2,\beta}\Big(\left(\begin{array}{l} 1\ \beta \\ 1\ \beta \\ 1 \end{array} \right| = 2\ \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array} \right| ,$$

$$D_{2,\beta}D_{1,\beta}\Big(\left(\begin{array}{l} \beta\ 2 \\ 1\ \beta \\ 1 \end{array} \right| + \left(\begin{array}{l} 1\ \beta \\ \beta\ 2 \\ 1 \end{array} \right| + \left(\begin{array}{l} 1\ \beta \\ 1\ 2 \\ \beta \end{array} \right| + \left(\begin{array}{l} 1\ 2 \\ 1\ \beta \\ \beta \end{array} \right| \Big) = 4\ \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array} \right| .$$

Therefore,

$$\begin{aligned} \Big((2!)^{-1}\ D_{1,\beta}D_{1,\beta}D_{\beta,1}D_{\beta,1} + (2!)^{-1}\ D_{2,\beta}D_{2,\beta}D_{\beta,2}D_{\beta,2}+ \\ + (2!)^{-1}\ D_{3,\beta}D_{3,\beta}D_{\beta,3}D_{\beta,3} + D_{2,\beta}D_{1,\beta}D_{\beta,1}D_{\beta,2}+ \\ + D_{3,\beta}D_{1,\beta}D_{\beta,1}D_{\beta,3} + D_{3,\beta}D_{2,\beta}D_{\beta,2}D_{\beta,3}\Big) \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array} \right| = 12\ \left(\begin{array}{l} 1\ 2 \\ 1\ 2 \\ 1 \end{array} \right| . \end{aligned}$$

Notice that, since $\widetilde{\lambda} = (3,2)$, according to Theorem 4.25, eq. 39, we have

$$h_2^*((3,2)) = \sum_{1 \le i_1 \le i_2 \le 3} (\widetilde{\lambda}_{i_1}+2-1)(\widetilde{\lambda}_{i_2}) = (3-2+1)2+(3-2+1)3+(2-2+1)2 = 12.$$

Moreover, by comparing with Example 4.8, since $\lambda = (2,2,1) = \widetilde{\mu}$, $\mu = (3,2)$, we have

$$e_2^*((2,2,1)) = 12 = h_2^*((3,2)). \tag{40}$$

$\square$

Proposition 4.27. *The set*

$$\big\{\mathbf{I}_1(n), \mathbf{I}_2(n), \dots, \mathbf{I}_n(n)\big\}$$

is a set of algebraically independent generators of the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$.

Given a shape

$$\lambda = (\lambda_1 \ge \cdots \ge \lambda_p),\ \ \lambda_1 \le n,$$

we set

$$\mathbf{I}_\lambda(n) = \mathbf{I}_{\lambda_1}(n)\mathbf{I}_{\lambda_2}(n)\cdots\mathbf{I}_{\lambda_p}(n).$$

By convention, if λ is the empty partition, we set $\mathbf{I}_\emptyset(n) = \mathbf{1} \in \boldsymbol{\zeta}(n)$.

4.3 The Shaped Capelli determinantal elements $\mathbf{K}_\lambda(n)$

Let $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\lambda_1 \leq n$. We notice that any element

$$e_{S_1, C_\lambda^*} \cdot e_{C_\lambda^*, S_2} \in Virt(m_0 + m_1, n),$$

where S_1, S_2 are tableaux on the proper alphabet $L = \{x_1, \ldots, x_n\}$ of shape λ, $\lambda_1 \leq n$, $m_0 \geq \widetilde{\lambda}_1$, is *skew-symmetric* in the rows of S_1 and S_2, respectively.

Definition 4.28. *We set*

$$\mathbf{K}_\lambda(n) = \sum_S \ \mathfrak{p}\big(e_{S, C_\lambda^*} \cdot e_{C_\lambda^*, S}\big) = \sum_S \ SC_\lambda^* \ C_\lambda^* S \in \mathbf{U}(gl(n)),$$

where the sum is extended to all row-increasing tableaux S on the proper alphabet $L = \{x_1, \ldots, x_n\}$.

By convention, if λ is the empty partition, we set $\mathbf{K}_\emptyset(n) = \mathbf{1} \in \boldsymbol{\zeta}(n)$.

Proposition 4.29. *Since the adjoint representation acts by derivation, we have*

$$ad(e_{ij})\big(\sum_S \ e_{S, C_\lambda^*} \cdot e_{C_\lambda^*, S}\big) = 0,$$

for every $e_{ij} \in gl(n)$.

Since the elements

$$e_{S_1, C_\lambda^*} \cdot e_{C_\lambda^*, S_2} \in Virt(m_0 + m_1, n),$$

are *skew-symmetric* in the rows of S_1 and S_2, the proof is essentially the same as the proof of Proposition 4.3.

From Remark 4.1, it follows

Theorem 4.30. *The elements $\mathbf{K}_\lambda(n)$ are central in $\mathbf{U}(gl(n))$.*

Example 4.31. Consider the action of $ad(e_{21})$ on $\mathbf{K}_{(2,1)}(2)$. We have

$$\begin{aligned}
ad(e_{21})\big(\mathbf{K}_{(2,1)}(2)\big) = &\mathfrak{p}\Big(+ e_{\substack{22\\1}\ \substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\ \substack{12\\1}} \\
&\quad + e_{\substack{12\\2}\ \substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\ \substack{12\\1}} \\
&\quad - e_{\substack{12\\1}\ \substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\ \substack{11\\1}} \Big) \\
+&\mathfrak{p}\Big(+ e_{\substack{22\\2}\ \substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\ \substack{12\\2}} \\
&\quad - e_{\substack{12\\2}\ \substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\ \substack{11\\2}} \\
&\quad - e_{\substack{12\\2}\ \substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\ \substack{12\\1}} \Big) = 0.
\end{aligned}$$

Consider the action of $ad(e_{12})$ on $\mathbf{K}_{(2,1)}(2)$. We have

$$\begin{aligned}
ad(e_{12})\big(\mathbf{K}_{(2,1)}(2)\big) = & \mathfrak{p}\Big(+ e_{\substack{11\\1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\1}} \\
& - e_{\substack{12\\1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{22\\1}} \\
& - e_{\substack{12\\1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\2}} \Big) \\
& + \mathfrak{p}\Big(+ e_{\substack{11\\2}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\2}} \\
& + e_{\substack{12\\1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\2}} \\
& - e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{22\\2}} \Big) = 0.
\end{aligned}$$

Consider the action of $ad(e_{11})$ on $\mathbf{K}_{(2,1)}(2)$. We have

$$\begin{aligned}
ad(e_{11})\big(\mathbf{K}_{(2,1)}(2)\big) = & \mathfrak{p}\Big(+ 2e_{\substack{12\\1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\1}} \\
& - 2e_{\substack{12\\1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\1}} \Big) \\
& + \mathfrak{p}\Big(+ e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\2}} \\
& - e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\2}} \Big) = 0.
\end{aligned}$$

Consider the action of $ad(e_{22})$ on $\mathbf{K}_{(2,1)}(2)$. We have

$$\begin{aligned}
ad(e_{22})\big(\mathbf{K}_{(2,1)}(2)\big) = & \mathfrak{p}\Big(+ e_{\substack{12\\1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\1}} \\
& - e_{\substack{12\\1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\1}} \Big) \\
& + \mathfrak{p}\Big(+ 2e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\2}} \\
& - 2e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\2}} \Big) = 0.
\end{aligned}$$

$\square$

Clearly,

$$\mathbf{K}_\lambda(n) \in \boldsymbol{\zeta}(n)^{(m)},$$

for every $m \geq |\lambda|$.

The elements $\mathbf{K}_\lambda(n)$ will be called *shaped Capelli determinantal elements.*

Theorem 4.32. *(Triangularity of the actions on highest weight vectors) Let $\lhd$ denote the dominance order on partitions. We have:*

If $\lvert\mu\rvert < \lvert\lambda\rvert$, then	$\mathbf{K}_\lambda(n)(v_{\widetilde{\mu}}) = 0,$	(41)
If $\lvert\mu\rvert = \lvert\lambda\rvert$ and $\mu \not\trianglerighteq \lambda$, then	$\mathbf{K}_\lambda(n)(v_{\widetilde{\mu}}) = 0,$	(42)
If $\lambda = \mu$, then	$\mathbf{K}_\lambda(n)(v_{\widetilde{\lambda}}) = (-1)^{\binom{k}{2}} H(\lambda) \cdot v_{\widetilde{\lambda}},$	(43)

where $H(\lambda) = H(\widetilde{\lambda})$ denotes the hook coefficient of the shape $\lambda \vdash k$.

Assertions (41) and (42) follow from the Proposition 3.37, eq. (23) and eq. (24), respectively; assertion (43) follows from Proposition 3.38.

Therefore, the central elements $\mathbf{K}_\lambda(n)$ are linearly independent, and the next result follows at once.

Proposition 4.33. *For every $m \in \mathbb{Z}^+$, the set*

$$\{\mathbf{K}_\lambda(n); \lambda_1 \leq n, \ |\lambda| \leq m \}$$

is a linear basis of $\boldsymbol{\zeta}(n)^{(m)}$.

The set

$$\{\mathbf{K}_\lambda(n); \lambda_1 \leq n \}$$

is a linear basis of the center $\boldsymbol{\zeta}(n)$.

The elements $\mathbf{K}_\lambda(n)$ are related to the Capelli generators $\mathbf{H}_k(n)$ by a *Straightenig Law.*

Proposition 4.34.

$$\mathbf{K}_\lambda(n) = \pm\mathbf{H}_\lambda(n) + \sum \ c_{\lambda,\mu}\mathbf{K}_\mu(n), \tag{44}$$

where $c_{\lambda,\mu} = 0$ unless $|\mu| \not< |\lambda|$.

Proof. From the virtual presentations of $\mathbf{K}_\lambda(n)$ and $\mathbf{H}_\lambda(n)$, it is easy to see - just applying the commutator relations in the algebra $\mathbf{U}(gl(m_0|m_1+n))$ - that the difference between the l.h.s. and the first summand of the r.h.s. in eq. (44) is a central element in $\boldsymbol{\zeta}(n)$ that belongs to a filtration element $\boldsymbol{\zeta}(n)^{(m)}$ with $m < |\lambda|$. □

4.4 The Shaped Nazarov/Umeda permanental elements $\mathbf{J}_\lambda(n)$

Given a column-nondecreasing tableaux S on the proper alphabet $L = \{x_1, \ldots, x_n\}$, let o_S denote the product of the factorials of the repetitions (of the symbols) column by column. For example, let

$$S = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 2 & 1 & 4 \\ 3 & 2 & 5 \\ 3 & 3 & \end{pmatrix}, \quad sh(S) = (3,3,3,3,2);$$

then $o_S = 2!\ 2!\ 3!\ 3!$.

Let $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\lambda \vdash d, \quad d \in \mathbb{Z}^+$.

Definition 4.35. *We set*

$$\mathbf{J}_\lambda(n) = \sum_S \ (o_S)^{-1} \ \mathfrak{p}\big(e_{S,D^*_{\widetilde{\lambda}}} \cdot e_{D^*_{\widetilde{\lambda}},S}\big) = \sum_S \ (o_S)^{-1} \ SD^*_{\widetilde{\lambda}} \ D^*_{\widetilde{\lambda}}S \in \mathbf{U}(gl(n)),$$

where the sum is extended to all column-nondecreasing tableaux S (of shape $\widetilde{\lambda}$) on the proper alphabet $L = \{x_1, \ldots, x_n\}$.

By convention, if λ is the empty partition, we set $\mathbf{J}_\emptyset(n) = \mathbf{1} \in \boldsymbol{\zeta}(n)$.

Claim 4.36. *We have*

$$e_{S,D^*_{\widetilde{\lambda}}} = e_{\widetilde{S},\widetilde{D^*_{\widetilde{\lambda}}}}$$

and

$$e_{D^*_{\widetilde{\lambda}},S} = e_{\widetilde{D^*_{\widetilde{\lambda}}},\widetilde{S}}.$$

Therefore

$$\mathbf{J}_\lambda(n) = \sum_T \ (o_{\widetilde{T}})^{-1} \ \mathfrak{p}\big(e_{T,\widetilde{D^*_{\widetilde{\lambda}}}} \cdot e_{\widetilde{D^*_{\widetilde{\lambda}}},T}\big) \tag{45}$$

$$= \sum_T \ (o_{\widetilde{T}})^{-1} \ T\widetilde{D^*_{\widetilde{\lambda}}} \ \widetilde{D^*_{\widetilde{\lambda}}}T \in \mathbf{U}(gl(n)), \tag{46}$$

where the sum is extended to all row-nondecreasing tableaux T (of shape λ) on the proper alphabet $L = \{x_1, \ldots, x_n\}$.

In particular, if $\lambda = (k)$, the row shape of length k, then $\mathbf{J}_\lambda(n) = \mathbf{I}_k(n)$.

Proposition 4.37. *Since the adjoint representation acts by derivation, we have*

$$ad(e_{ij})\big(\sum_S \ (o_S)^{-1} \ e_{S,D^*_{\widetilde{\lambda}}} \cdot e_{D^*_{\widetilde{\lambda}},S}\big) = 0,$$

for every $e_{ij} \in gl(n)$.

The proof is essentially the same as the proof of Proposition 4.23.

From Remark 4.1, it follows

Theorem 4.38. *The elements $\mathbf{J}_\lambda(n)$ are central in $\mathbf{U}(gl(n))$.*

Hence, the elements $\mathbf{J}_\lambda(n)$ are central in $\mathbf{U}(gl(n))$, by Remark 4.1.

Clearly,

$$\mathbf{J}_\lambda(n) \in \boldsymbol{\zeta}(n)^{(m)},$$

for every $m \geq |\lambda|$.

The elements $J_\lambda(n)$ will be called *shaped Nazarov/Umeda permanental elements.*

Proposition 4.39. *(Triangularity of the actions on highest weight vectors) We have:*

$$\text{If } |\mu| < |\lambda|, \text{ then} \qquad \mathbf{J}_\lambda(n)(v_{\widetilde{\mu}}) = 0, \tag{47}$$

$$\text{If } |\mu| = |\lambda| \text{ and } \widetilde{\mu} \not\trianglerighteq \lambda, \text{ then} \qquad \mathbf{J}_\lambda(n)(v_{\widetilde{\mu}}) = 0. \tag{48}$$

Assertions (47) and (48) follow from the Proposition 3.37, eq. (22) and eq. (25), respectively.

Proposition 4.40. *For every shape λ, we have:*

$$\mathbf{J}_\lambda(n) = \mathbf{I}_\lambda(n) + \mathbf{F}_\lambda(n), \tag{49}$$

where

$$\mathbf{I}_\lambda(n) \notin \boldsymbol{\zeta}(n)^{(m)}, \quad \textit{if } m < |\lambda|,$$
$$\mathbf{F}_\lambda(n) \in \boldsymbol{\zeta}(n)^{(p)}, \quad \textit{for some } p < |\lambda|.$$

Proof. From the virtual presentations of $\mathbf{J}_\lambda(n)$ and $\mathbf{I}_\lambda(n)$, it is easy to see - just applying the commutator relations in the algebra $\mathbf{U}(gl(m_0|m_1+n))$ - that the difference between the l.h.s. and the first summand of the r.h.s. in eq. (49) is a central element in $\boldsymbol{\zeta}(n)$ that belongs to a filtration element $\boldsymbol{\zeta}(n)^{(m)}$ with $m < |\lambda|$. ☐

Example 4.41. In order to simplify the notation, we will write any summand $e_{T,\widetilde{D^*_{\tilde\lambda}}} \cdot e_{\widetilde{D^*_{\tilde\lambda}},T}$ in eq. (45) as $[T|T]^*$. Let $\lambda = (2,2)$, and $n = 2$. In this notation, the central element $\mathbf{J}_{(2,2)}(2)$ can be written in the following way:

$$\begin{aligned}
\mathbf{J}_{(2,2)}(2) = {} & \frac{1}{2!2!}\left[\begin{array}{c|c} 1\,1 & 1\,1 \\ 1\,1 & 1\,1 \end{array}\right]^* + \frac{1}{2!}\left[\begin{array}{c|c} 1\,1 & 1\,1 \\ 1\,2 & 1\,2 \end{array}\right]^* + \frac{1}{2!2!}\left[\begin{array}{c|c} 1\,1 & 1\,1 \\ 2\,2 & 2\,2 \end{array}\right]^* + \\
& + \frac{1}{2!}\left[\begin{array}{c|c} 1\,2 & 1\,2 \\ 1\,1 & 1\,1 \end{array}\right]^* + \left[\begin{array}{c|c} 1\,2 & 1\,2 \\ 1\,2 & 1\,2 \end{array}\right]^* + \frac{1}{2!}\left[\begin{array}{c|c} 1\,2 & 1\,2 \\ 2\,2 & 2\,2 \end{array}\right]^* + \\
& + \frac{1}{2!2!}\left[\begin{array}{c|c} 2\,2 & 2\,2 \\ 1\,1 & 1\,1 \end{array}\right]^* + \frac{1}{2!}\left[\begin{array}{c|c} 2\,2 & 2\,2 \\ 1\,2 & 1\,2 \end{array}\right]^* + \frac{1}{2!2!}\left[\begin{array}{c|c} 2\,2 & 2\,2 \\ 2\,2 & 2\,2 \end{array}\right]^*.
\end{aligned}$$

In turn, applying the commutator relations, we get:

$$\begin{aligned}
\mathbf{J}_{(2,2)}(2) = {} & \frac{1}{2!}[11|11]^*\frac{1}{2!}[11|11]^* + \mathbf{G}_1 + \frac{1}{2!}[11|11]^*[12|12]^* + \mathbf{G}_2 + \\
& + \frac{1}{2!}[11|11]^*\frac{1}{2!}[11|22]^* + \mathbf{G}_3 + [12|12]^*\frac{1}{2!}[11|11]^* + \mathbf{G}_4 + \\
& + [12|12]^*[12|12]^* + \mathbf{G}_5 + [12|12]^*\frac{1}{2!}[22|22]^* + \mathbf{G}_6 + \\
& + \frac{1}{2!}[22|22]^*\frac{1}{2!}[11|11]^* + \mathbf{G}_7 + \frac{1}{2!}[22|22]^*[12|12]^* + \mathbf{G}_8 + \\
& + \frac{1}{2!}[22|22]^*\frac{1}{2!}[22|22]^* + \mathbf{G}_9,
\end{aligned}$$

where $\mathbf{G}_j \in \boldsymbol{U}(gl(2))^{(p)}$, for some $p < |\lambda| = 4$, for $j = 1, 2, \ldots, 9$. Therefore,

$$\begin{aligned}
\mathbf{J}_{(2,2)}(2) &= \Big(\frac{1}{2!}[11|11]^* + [12|12]^* + \frac{1}{2!}[22|22]^*\Big)^2 + \sum_{j=1}^{9}\ \mathbf{G}_j = \mathbf{I}_2(2)^2 + \sum_{j=1}^{9}\ \mathbf{G}_j \\
&= \mathbf{I}_{(2,2)}(2) + \sum_{j=1}^{9}\ \mathbf{G}_j,
\end{aligned}$$

where $\mathbf{I}_{(2,2)}(2) \notin \boldsymbol{\zeta}(2)^{(m)}$, if $m < |\lambda| = 4$, and $\sum_{j=1}^{9}\ \mathbf{G}_j \in \boldsymbol{\zeta}(2)^{(p)}$, for some $p < |\lambda| = 4$.

$\square$

Proposition 4.42. *For every $m \in \mathbb{Z}^+$, the set*

$$\left\{ \mathbf{J}_\lambda(n);\ \lambda_1 \le n,\ |\lambda| \le m \right\}$$

is a linear basis of $\boldsymbol{\zeta}(n)^{(m)}$.

The set

$$\left\{ \mathbf{J}_\lambda(n);\ \lambda_1 \le n \right\}$$

is a linear basis of the center $\boldsymbol{\zeta}(n)$.

Proof. We first notice that the elements of the set $\left\{ \mathbf{J}_\lambda(n);\ \lambda_1 \le n \right\}$ are linearly independent. Indeed, by Proposition 4.40, we can limit ourselves to consider a linear combination

$$\sum_\lambda c_\lambda\, \mathbf{J}_\lambda(n), \quad \lambda_1 \le n,\ |\lambda| = m.$$

If this linear combination equals zero, then, again by Proposition 4.40, we should have $\sum_\lambda c_\lambda\, \mathbf{I}_\lambda(n) = 0$; since each $\mathbf{I}_\lambda(n)$ is a product of Nazarov/Umeda elements $\mathbf{I}_k(n)$, $k \le n$, this is a contradiction with respect to Proposition 4.27. Since the two sets

$$\left\{ \mathbf{J}_\lambda(n);\ \lambda_1 \le n,\ |\lambda| \le m \right\} \text{ and } \left\{ \mathbf{K}_\lambda(n);\ \lambda_1 \le n,\ |\lambda| \le m \right\}$$

have the same cardinality, the assertions follow from Proposition 4.33. $\square$

By Proposition 4.42, we get a stronger version of Proposition 4.40.

Proposition 4.43. *We have*

$$\mathbf{J}_\lambda(n) = \mathbf{I}_\lambda(n) + \sum d_{\lambda,\mu} \mathbf{J}_\mu(n),$$

where $d_{\lambda,\mu} = 0$ unless $|\mu| < |\lambda|$.

Example 4.44. Consider the Harish-Chandra isomorphism

$$\chi_2 : \boldsymbol{\zeta}(2) \to \Lambda^*(2),$$

where $\Lambda^*(2)$ denotes the algebra of *shifted symmetric* polynomials in two variables [56]. By computing the eigevalues of highest weight vectors as polynomials in the weight, we have:

$$\begin{aligned}
\chi_2(\mathbf{J}_{(2,2)}(2)) &= y^4 + 2xy^3 - 8y^3 + 3x^2y^2 - 11xy^2 + 21y^2 + \\
&\qquad + 2x^3y - 11x^2y + 19xy - 18y + x^4 - 6x^3 + 11x^2 - 6x, \\
\chi_2(\mathbf{J}_{(2,1)}(2)) &= y^3 + 2xy^2 - 4y^2 + 2x^2y - 5xy + 4y + x^3 - 3x^2 + 2x, \\
\chi_2(\mathbf{J}_{(2)}(2)) &= \chi_2(\mathbf{I}_2(2)) = y^2 + xy - 2y + x^2 - x, \\
\chi_2(\mathbf{J}_{(1,1,1)}(2)) &= y^3 + 3xy^2 - 3y^2 + 3x^2y - 6xy + 2y + x^3 - 3x^2 + 2x, \\
\chi_2(\mathbf{I}_1(2)) &= y + x.
\end{aligned}$$

By direct computations, we get the following identity in the algebra $\Lambda^*(2)$:

$$\chi_2\big(\mathbf{J}_{(2,2)}(2)\big) = \chi_2\big(\mathbf{I}_2(2)^2 - 7\mathbf{J}_{(2,1)}(2) - 2\mathbf{J}_{(2)}(2) + 3\mathbf{J}_{(1,1,1)}(2)\big)$$
$$= \chi_2\big(\mathbf{I}_2(2)^2 - 7\mathbf{I}_2(2)\mathbf{I}_1(2) + 3\mathbf{I}_1(2)^3 + 12\mathbf{I}_2(2) - 9\mathbf{I}_1(2)^2 + 6\mathbf{I}_1(2)\big).$$

Hence, in the center $\boldsymbol{\zeta}(2)$, we have:

$$\mathbf{J}_{(2,2)}(2) = \mathbf{I}_2(2)^2 - 7\mathbf{J}_{(2,1)}(2) - 2\mathbf{J}_{(2)}(2) + 3\mathbf{J}_{(1,1,1)}(2),$$
$$= \mathbf{I}_2(2)^2 - 7\mathbf{I}_2(2)\mathbf{I}_1(2) + 3\mathbf{I}_1(2)^3 + 12\mathbf{I}_2(2) - 9\mathbf{I}_1(2)^2 + 6\mathbf{I}_1(2),$$

(Notice that $\mathbf{I}_2(2)^2 = \mathbf{I}_{(2,2)}(2)$).

4.5 The virtual form of the Schur/Okounkov/Sahi basis $\mathbf{S}_\lambda(n)$

4.5.1 The virtual definition of $\mathbf{S}_\lambda(n)$ and main results

Let $\lambda \vdash d$ be a partition, $\lambda_1 \leq n$. We notice that any element

$$e_{S_1,C^*_\lambda} \cdot e_{C^*_\lambda,D^*_\lambda} \cdot e_{D^*_\lambda,C^*_\lambda} \cdot e_{C^*_\lambda,S_2} \in Virt(m_0 + m_1, n),$$

where S_1, S_2 are tableaux on the proper alphabet $L = \{x_1, \ldots, x_n\}$ of shape λ, $\lambda_1 \leq n$, $m_0 \geq \widetilde{\lambda}_1$, $m_1 \geq \lambda_1$, is *skew-symmetric* in the rows of S_1 and S_2, respectively.

Definition 4.45. *We set*

$$\mathbf{S}_\lambda(n) = \frac{1}{H(\lambda)} \sum_S \mathfrak{p}\big(e_{S,C^*_\lambda} \cdot e_{C^*_\lambda,D^*_\lambda} \cdot e_{D^*_\lambda,C^*_\lambda} \cdot e_{C^*_\lambda,S}\big)$$
$$= \frac{1}{H(\lambda)} \sum_S SC^*_\lambda \; C^*_\lambda D^*_\lambda \; D^*_\lambda C^*_\lambda \; C^*_\lambda S \in \mathbf{U}(gl(n)),$$

where the sum is extended to all row (strictly) increasing tableaux S on the proper alphabet $L = \{x_1, \ldots, x_n\}$.

By convention, if λ is the empty partition, we set $\mathbf{S}_\emptyset(n) = \mathbf{1} \in \boldsymbol{\zeta}(n)$.

The element $\mathbf{S}_\lambda(n) \in \mathbf{U}(gl(n))$ is called the *Schur element* of shape λ in dimension n.

Example 4.46. Let $\lambda = (2,1) \vdash d = 3$, $n = 2$. The Schur element $\mathbf{S}_{(2,1)}(2) \in \mathbf{U}(gl(2))$ is expressed as the image (under the Capelli epimorphism $\mathfrak{p}$) of the sum of three monomials in $Virt(m_0 + m_1, 3) \subset \mathbf{U}(gl(m_0|m_1 + 2))$, $m_0 \geq 2$ and $m_1 \geq$:

$$\mathbf{S}_{(2,1)}(2) = \frac{1}{3} \times$$
$$\mathfrak{p}\Big(e_{1\alpha_1}e_{2\alpha_1}e_{1\alpha_2}e_{\alpha_1\beta_1}e_{\alpha_1\beta_2}e_{\alpha_2\beta_1}e_{\beta_1\alpha_1}e_{\beta_2\alpha_1}e_{\beta_1\alpha_2}e_{\alpha_1 1}e_{\alpha_1 2}e_{\alpha_2 1} +$$
$$+ e_{1\alpha_1}e_{2\alpha_1}e_{2\alpha_2}e_{\alpha_1\beta_1}e_{\alpha_1\beta_2}e_{\alpha_2\beta_1}e_{\beta_1\alpha_1}e_{\beta_2\alpha_1}e_{\beta_1\alpha_2}e_{\alpha_1 1}e_{\alpha_1 2}e_{\alpha_2 2}\Big),$$

where $\alpha_1, \alpha_2 \in A_0$ and $\beta_1, \beta_2, \in A_1$.

In the notation and terminology of Subsection 3.6.2, the Schur element $\mathbf{S}_{(2,1)}(2)$ is expressed as the image $\mathfrak{p}$ as the sum of two elements of $\mathbf{U}(gl(2))$, each of them being the product of four *bitableau monomials*:

$$\begin{aligned}\mathbf{S}_{(2,1)}(2) = \frac{1}{3} \times \mathfrak{p}\Big(e_{\substack{12\\1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{\beta_1\beta_2\\ \beta_1}} \times e_{\substack{\beta_1\beta_2\\ \beta_1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{12\\1}}\Big) + \\ + \frac{1}{3} \times \mathfrak{p}\Big(e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{\beta_1\beta_2\\ \beta_1}} \times e_{\substack{\beta_1\beta_2\\ \beta_1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{12\\2}}\Big).\end{aligned}$$

Therefore, the Schur element $\mathbf{S}_{(2,1)}(2)$ acts on the Schur module $Schur_\mu(2)$, $\mu_1 \leq 2$, by the following polynomial in superpolarization operators:

$$\begin{aligned}\frac{1}{3} \times (D_{1\alpha_1} D_{2\alpha_1} D_{1\alpha_2} D_{\alpha_1\beta_1} D_{\alpha_1\beta_2} D_{\alpha_2\beta_1} D_{\beta_1\alpha_1} D_{\beta_2\alpha_1} D_{\beta_1\alpha_2} D_{\alpha_1 1} D_{\alpha_1 2} D_{\alpha_2 1} + \\ + D_{1\alpha_1} D_{2\alpha_1} D_{2\alpha_2} D_{\alpha_1\beta_1} D_{\alpha_1\beta_2} D_{\alpha_2\beta_1} D_{\beta_1\alpha_1} D_{\beta_2\alpha_1} D_{\beta_1\alpha_2} D_{\alpha_1 1} D_{\alpha_1 2} D_{\alpha_2 2}).\end{aligned}$$

□

Proposition 4.47. *Consider the element*

$$\sum_S e_{S,C^*_\lambda} \cdot e_{C^*_\lambda,D^*_\lambda} \cdot e_{D^*_\lambda,C^*_\lambda} \cdot e_{C^*_\lambda,S},$$

where the sum is extended to all row (strictly) increasing tableaux S on the proper alphabet $L = \{x_1, \ldots, x_n\}$.

Since the adjoint representation acts by derivation, we have

$$ad(e_{ij})\big(\sum_S e_{S,C^*_\lambda} \cdot e_{C^*_\lambda,D^*_\lambda} \cdot e_{D^*_\lambda,C^*_\lambda} \cdot e_{C^*_\lambda,S}\big) = 0,$$

for every $e_{ij} \in gl(n)$.

The proof is essentially the same as the proof of Proposition 4.29.

Example 4.48. Consider the action of $ad(e_{21})$ on $\mathbf{S}_{(2,1)}(2)$. We have

$$\begin{aligned}ad(e_{21})\big(\mathbf{S}_{(2,1)}(2)\big) = & \frac{1}{3} \times \mathfrak{p}\Big(+ e_{\substack{22\\1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{\beta_1\beta_2\\ \beta_1}} \times e_{\substack{\beta_1\beta_2\\ \beta_1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{12\\1}} \\ & + e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{\beta_1\beta_2\\ \beta_1}} \times e_{\substack{\beta_1\beta_2\\ \beta_1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{12\\1}} \\ & - e_{\substack{12\\1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{\beta_1\beta_2\\ \beta_1}} \times e_{\substack{\beta_1\beta_2\\ \beta_1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{11\\1}}\Big) \\ & + \frac{1}{3} \times \mathfrak{p}\Big(+ e_{\substack{22\\2}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{\beta_1\beta_2\\ \beta_1}} \times e_{\substack{\beta_1\beta_2\\ \beta_1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{12\\2}} \\ & - e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{\beta_1\beta_2\\ \beta_1}} \times e_{\substack{\beta_1\beta_2\\ \beta_1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{11\\2}} \\ & - e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{\beta_1\beta_2\\ \beta_1}} \times e_{\substack{\beta_1\beta_2\\ \beta_1}\,\substack{\alpha_1\alpha_1\\ \alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\ \alpha_2}\,\substack{12\\1}}\Big),\end{aligned}$$

that equals, by row skew-symmetry,

$$ad(e_{21})\big(\mathbf{S}_{(2,1)}(2)\big) = \frac{1}{3} \times \mathfrak{p}\Big(+ e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{\beta_1\beta_2\\\beta_1}} \times e_{\substack{\beta_1\beta_2\\\beta_1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\1}}$$
$$- e_{\substack{12\\2}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{\beta_1\beta_2\\\beta_1}} \times e_{\substack{\beta_1\beta_2\\\beta_1}\,\substack{\alpha_1\alpha_1\\\alpha_2}} \times e_{\substack{\alpha_1\alpha_1\\\alpha_2}\,\substack{12\\1}}\Big) = 0.$$

$\square$

From Remark 4.1, it follows

Theorem 4.49. *The Schur elements* $\mathbf{S}_\lambda(n)$ *are central in* $\mathbf{U}(gl(n))$.

Clearly,

$$\mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}(n)^{(m)}, \tag{50}$$

for every $m \geq |\lambda|$.

Theorem 4.50. *(Vanishing theorem) We have:*

$$\begin{array}{ll} \text{If } \lambda \not\subseteq \mu, \text{ then} & \mathbf{S}_\lambda(n)(v_{\tilde{\mu}}) = 0, \\ \text{If } \mu = \lambda, \text{ then} & \mathbf{S}_\lambda(n)(v_{\tilde{\lambda}}) = H(\lambda) \cdot v_{\tilde{\lambda}}, \end{array}$$

where $H(\lambda)$ *denotes the hook number of the shape (partition)* $\lambda \vdash k$.

Proof. The first assertion is an immediate consequence of Proposition 3.37, eq. (28).

Recall that the element

$$v_{\tilde{\lambda}} = (D_\lambda | D_\lambda^P)$$

is the "canonical" highest weight vector of the irreducible $gl(n)-$module $Schur_\lambda(n)$. Clearly

$$\begin{aligned} \mathbf{S}_\lambda(n)\left(v_{\tilde{\lambda}}\right) &= \frac{1}{H(\lambda)} \sum_S SC_\lambda^* \; C_\lambda^* D_\lambda^* \; D_\lambda^* C_\lambda^* \; C_\lambda^* S\big((D_\lambda|D_\lambda^P)\big) \\ &= \frac{1}{H(\lambda)} \; D_\lambda C_\lambda^* \; C_\lambda^* D_\lambda^* \; D_\lambda^* C_\lambda^* \; C_\lambda^* D_\lambda\big((D_\lambda|D_\lambda^P)\big). \end{aligned}$$

By Proposition 3.38, eqs. (30), (32),

$$\begin{aligned} \mathbf{S}_\lambda(n)\left(v_{\tilde{\lambda}}\right) &= \frac{1}{H(\lambda)}(-1)^{\binom{k}{2}} \; H(\lambda) \cdot D_\lambda C_\lambda^* \; C_\lambda^* D_\lambda^* \; D_\lambda^* C_\lambda^*\big((C_\lambda^*|D_\lambda^P) \; (\lambda!)^{-1}\big) \\ &= \frac{1}{H(\lambda)}(-1)^{\binom{k}{2}} \; H(\lambda) \cdot D_\lambda C_\lambda^* \; C_\lambda^* D_\lambda^*\big((D_\lambda^*|D_\lambda^P)\big) \\ &= \frac{1}{H(\lambda)}(-1)^{\binom{k}{2}} \; H(\lambda) \cdot D_\lambda C_\lambda^*\big((C_\lambda^*|D_\lambda^P) \; (\lambda!)^{-1}\big) \cdot H(\lambda)(-1)^{\binom{k}{2}} \\ &= H(\lambda) \cdot (D_\lambda|D_\lambda^P). \end{aligned}$$

$\square$

Theorem 4.51. *(Triangularity/orthogonality of the actions on highest weight vectors) We have:*

$$\begin{array}{ll} \text{If } |\mu| < |\lambda|, \text{ then} & \mathbf{S}_\lambda(n)(v_{\widetilde{\mu}}) = 0, \\ \text{If } |\mu| = |\lambda|, \text{ then} & \mathbf{S}_\lambda(n)(v_{\widetilde{\mu}}) = \delta_{\lambda,\mu} \cdot H(\lambda) \cdot v_{\widetilde{\mu}}. \end{array}$$

Proof. The first assertion is an immediate consequence of Proposition 3.37, eq. (23). The fact that, if $|\mu| = |\lambda|$, $\mu \neq \lambda$, then $\mathbf{S}_\lambda(n)(v_{\widetilde{\mu}}) = 0$, is an immediate consequence of Proposition 3.37, eq. (26). □

Theorem 4.52. *For every $m \in \mathbb{Z}^+$, the set*

$$\{\mathbf{S}_\lambda(n);\ \lambda_1 \leq n,\ |\lambda| \leq m\ \}$$

is a linear basis of $\boldsymbol{\zeta}(n)^{(m)}$.

The set

$$\{\mathbf{S}_\lambda(n);\ \lambda_1 \leq n\ \}$$

is a linear basis of the center $\boldsymbol{\zeta}(n)$.

4.5.2 The determinantal Capelli generators and the permanental Nazarov/Umeda elements as elements of the Schur basis

Proposition 4.53. *Let $\lambda = (k)$ denote the row shape of length k. We have*

$$\mathbf{S}_{(k)}(n) = \mathbf{H}_k(n).$$

Proof. We have

$$\begin{aligned} \mathbf{S}_{(k)}(n) &= \frac{1}{H(\lambda)} \sum_S \mathfrak{p}\big(e_{S,C^*_{(k)}} \cdot e_{C^*_{(k)},D^*_{(k)}} \cdot e_{D^*_{(k)},C^*_{(k)}} \cdot e_{C^*_{(k)},S}\big) \\ &= \frac{1}{k!} \sum_S\ SC^*_{(k)}\ C^*_{(k)}D^*_{(k)}\ D^*_{(k)}C^*_{(k)}\ C^*_{(k)}S, \end{aligned}$$

where the sum is extended to all strictly increasing row tableaux S of shape (k).

Notice that

$$\mathfrak{p}\big(e_{S,C^*_{(k)}} \cdot e_{C^*_{(k)},D^*_{(k)}} \cdot e_{D^*_{(k)},C^*_{(k)}} \cdot e_{C^*_{(k)},S}\big)$$

equals

$$(-1)^{\binom{k}{2}}\ \mathfrak{p}\big(e_{S,C^*_{(k)}} \cdot e_{C^*_{(k)},C^*_{(k)}} \cdot e_{C^*_{(k)},S}\big),$$

that, in turn, equals

$$(-1)^{\binom{k}{2}}\ k!\ \mathfrak{p}\big(e_{S,C^*_{(k)}} \cdot e_{C^*_{(k)},S}\big).$$

Therefore,

$$\begin{aligned} \mathbf{S}_{(k)}(n) = (-1)^{\binom{k}{2}} &\sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathfrak{p}\big(e_{i_1,\alpha} e_{i_2,\alpha} \cdots e_{i_k,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}\big) \\ = &\sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathfrak{p}\big(e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}\big) \\ = &\ \mathbf{H}_k(n). \end{aligned}$$

□

Remark 4.54. *Notice that $e_k^*((k))$, the eigenvalue of $\mathbf{H}_k$ on the row shape (k) of length k, equals the hook coefficient $H((k)) = k!$.*

Proposition 4.55. *Let $\lambda = (1^k)$ denote the column shape of length k. We have*

$$\mathbf{S}_{(1^k)}(n) = \mathbf{I}_k(n).$$

Proof. We have

$$\begin{aligned}\mathbf{S}_{(1^k)}(n) &= \frac{1}{H(\lambda)} \sum_S \mathfrak{p}\big(e_{S,C^*_{(1^k)}} \cdot e_{C^*_{(1^k)},D^*_{(1^k)}} \cdot e_{D^*_{(1^k)},C^*_{(1^k)}} \cdot e_{C^*_{(1^k)},S}\big) \\ &= \frac{1}{k!} \sum_S \; SC^*_{(1^k)} \; C^*_{(1^k)}D^*_{(1^k)} \; D^*_{(1^k)}C^*_{(1^k)} \; C^*_{(1^k)}S,\end{aligned}$$

where the sum is extended to all column tableaux S of shape (1^k).

Since the column tableau $C^*_{(1^k)}$ is multilinear (that is,

$$C^*_{(1^k)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix},$$

where the α_i's are distinct positive virtual symbols), then each summand

$$\mathfrak{p}\big(e_{S,C^*_{(1^k)}} \cdot e_{C^*_{(1^k)},D^*_{(1^k)}} \cdot e_{D^*_{(1^k)},C^*_{(1^k)}} \cdot e_{C^*_{(1^k)},S}\big)$$

equals

$$\mathfrak{p}\big(e_{S,D^*_{(1^k)}} \cdot e_{D^*_{(1^k)},S}\big).$$

Therefore

$$\begin{aligned}\mathbf{S}_{(1^k)}(n) &= \frac{1}{k!} \sum_S \; SD^*_{(1^k)}D^*_{(1^k)}S \\ &= \frac{1}{k!} \sum_{(h_1,\ldots,h_n)} \sum_T \; TD^*_{(1^k)}D^*_{(1^k)}T,\end{aligned}$$

where the outer sum is extended over all indexes $h_1+\cdots+h_n = k$ and inner sum is extended over all column tableaux T with h_1 occurrences of 1, h_2 occurrences of 2, …, h_n occurrences of n. Moreover, since each element $TD^*_{(1^k)}$ and $D^*_{(1^k)}T$ is row-commutative, then the inner sum

$$\sum_T \; TD^*_{(1^k)}D^*_{(1^k)}T$$

equals

$$\binom{k}{h_1, h_2, \ldots, h_n} \cdot \left[\begin{array}{c|c} 1 & \beta_1 \\ \vdots & \vdots \\ 1 & \beta_1 \\ \vdots & \vdots \\ n & \beta_1 \\ \vdots & \vdots \\ n & \beta_1 \end{array}\right] \left[\begin{array}{c|c} \beta_1 & 1 \\ \vdots & \vdots \\ \beta_1 & 1 \\ \vdots & \vdots \\ \beta_1 & n \\ \vdots & \vdots \\ \beta_1 & n \end{array}\right],$$

where there are h_1 occurrences of 1, h_2 occurrences of 2, ..., h_n occurrences of n. Therefore

$$\begin{aligned}
\mathbf{S}_{(1^k)}(n) &= \frac{1}{k!} \sum_{(h_1,\ldots,h_n)} \sum_{T} T D^*_{(1^k)} D^*_{(1^k)} T \\
&= \frac{1}{k!} \sum_{(h_1,\ldots,h_n)} \binom{k}{h_1, h_2, \ldots, h_n} \cdot \left[\begin{array}{c|c} 1 & \beta_1 \\ \vdots & \vdots \\ 1 & \beta_1 \\ \vdots & \vdots \\ n & \beta_1 \\ \vdots & \vdots \\ n & \beta_1 \end{array}\right] \left[\begin{array}{c|c} \beta_1 & 1 \\ \vdots & \vdots \\ \beta_1 & 1 \\ \vdots & \vdots \\ \beta_1 & n \\ \vdots & \vdots \\ \beta_1 & n \end{array}\right] \\
&= \sum_{(h_1,h_2,\ldots,h_n)} \frac{1}{h_1! h_2! \cdots h_n!} [n^{h_n} \cdots 2^{h_2} 1^{h_1} | 1^{h_1} 2^{h_2} \cdots n^{h_n}]^* \\
&= \mathbf{I}_k(n).
\end{aligned}$$

□

Remark 4.56. *Notice that $h_k^*((1^k))$, the eigenvalue of $\mathbf{I}_k$ on the column shape (1^k) of length k, equals the hook coefficient $H((1^k)) = k!$.*

4.5.3 The Sahi/Okounkov Characterization Theorem

We anticipate the formulation of Theorem 4.51 in terms of the Harish-Chandra isomorphism (Definition 6.2, below)

$$\chi_n : \boldsymbol{\zeta}(n) \longrightarrow \Lambda^*(n),$$

where $\Lambda^*(n)$ denotes the algebra of *shifted symmetric polynomials* in n variables (see Section 6 below).

Proposition 4.57. *We have that $\chi_n(\mathbf{S}_\lambda(n))$ is an element $m-$th filtration element $\Lambda^*(n)^{(m)}$, for any $m \geq |\lambda|$, of the algebra $\Lambda^*(n)$ of shifted symmetric* *polynomials in n variables. Furthermore*

$$\begin{array}{ll} \text{If } |\mu| < |\lambda|, \text{ then} & \chi_n(\mathbf{S}_\lambda(n))(\widetilde{\mu}) = 0, \\ \text{If } |\mu| = |\lambda|, \text{ then} & \chi_n(\mathbf{S}_\lambda(n))(\widetilde{\mu}) = \delta_{\lambda,\mu} \cdot H(\lambda), \end{array}$$

where $H(\lambda)$ denotes the hook number of the shape (partition) λ.

By combining Proposition 4.57 with the *Sahi/Okounkov Characterization Theorem* (Theorem 1 of [63] and Theorem 3.3 of [56], see also [54]), the shifted symmetric polynomial $\chi_n(\mathbf{S}_\lambda(n))$ is the *Schur shifted symmetric polynomial* $s^*_{\widetilde{\lambda}|n}$ of [56]. It follows that the basis $\{\mathbf{S}_\lambda(n);\ \lambda_1 \leq n\}$ of the center $\boldsymbol{\zeta}(n)$ is the preimage of the basis of Schur shifted symmetric polynomials of $\Lambda^*(n)$ characterized and described by Sahi [63] (*recursive procedure*), and, moreover, it coincides with the basis of $\boldsymbol{\zeta}(n)$ described by Okounkov in terms of *quantum immanants* [54] (for further descriptions, see also [55] and [53]).

Remark 4.58. *Notice that $e^*_k((k))$, the eigenvalue of $\mathbf{H}_k$ on the row shape (k) of length k, equals the hook coefficient $H((k))$. Similarly, $h^*_k((1^k))$, the eigenvalue of $\mathbf{I}_k$ on the column shape (1^k) of length k, equals the hook coefficient $H((1^k))$.*

4.6 Duality in $\boldsymbol{\zeta}(n)$

Let

$$\mathcal{W}_n : \boldsymbol{\zeta}(n) \to \boldsymbol{\zeta}(n)$$

be the algebra automorphism defined by setting

$$\mathcal{W}_n\Big(\mathbf{H}_k(n)\Big) = \mathbf{I}_k(n), \quad k = 1, 2, \ldots, n.$$

Clearly, Proposition 4.7 and Theorem 4.25.2 imply that, if $\mu_1, \widetilde{\mu}_1 \leq n$, then

$$e^*_k(\widetilde{\mu}) = h^*_k(\mu). \tag{51}$$

Theorem 4.59. *Let μ be such that $\mu_1, \widetilde{\mu}_1 \leq n$. For every $\boldsymbol{\varrho} \in \boldsymbol{\zeta}(n)$ the eigenvalue of $\boldsymbol{\varrho}$ on the $gl(n)-$irreducible module $Schur_\mu(n)$ (with highest weight $\widetilde{\mu}$) equals eigenvalue of $\mathcal{W}_n\Big(\boldsymbol{\varrho}\Big)$ on the $gl(n)-$irreducible module $Schur_{\widetilde{\mu}}(n)$ (with highest weight μ).*

Example 4.60. From Example 4.8, the eigenvalue of $\mathbf{H}_2(3)$ on the $gl(3)$-irreducible module $Schur_{(3,2)}(3)$ equals $e^*_2((2,2,1)) = 12$.

From Example 4.26, the eigenvalue of $\mathcal{W}_3\Big(\mathbf{H}_2(3)\Big) = \mathbf{I}_2(3)$ on the $gl(3)$-irreducible module $Schur_{(2,2,1)}(3)$ equals $h^*_2((3,2)) = 12$.

$\square$

The preceding result, in combination with the characterization Theorems of subsection 4.5.3, implies

Corollary 4.61. *Let* $\lambda_1, \widetilde{\lambda}_1 \leq n$. *Since* $H(\lambda) = H(\widetilde{\lambda})$, *then*

$$\mathcal{W}_n\Big(\mathbf{S}_\lambda(n)\Big) = \mathbf{S}_{\widetilde{\lambda}}(n).$$

Since the $\mathbf{H}_k(n)$'s and the $\mathbf{I}_k(n)$'s, $k = 1, 2, \ldots, n$, are elements of the Schur basis associated to pairs of (row/column) conjugate partitions that satisfy the conditions of Corollary 4.59 , then

$$\mathcal{W}_n\Big(\mathbf{I}_k(n)\Big) = \mathbf{H}_k(n), \quad k = 1, 2, \ldots, n.$$

Corollary 4.62. *The algebra automorphism* $\mathcal{W}_n$ *is an involution.*

Remark 4.63. *The preceding results admit a representation-theoretic interpretation.*

Given a partition μ, *there are two non equivalent functors (on the category of commutative ring with unity) that associate to* μ *a* $GL(n)-$*indecomposable representation, the Schur functor and the Weyl functor (see, e.g. [3], where the Weyl functor is called co-Schur functor). When evaluated on a field of characteristic zero, the two functors produce irreducible modules* $Schur_\mu(n)$ *and* $Weyl_\mu(n)$ *of highest weight* $\widetilde{\mu}$ *and* μ, *respectively. Hence* $Schur_\mu(n)$ *is isomorphic to* $Weyl_{\widetilde{\mu}}(n)$, *and then, passing to characters, the "duality" between Schur and Weyl modules can be regarded as a representation-theoretic version of the classical involution of the algebra* $\Lambda(n)$ *of symmetric polynomials. In this language, Theorem 4.59 can be restated by saying that the eigenvalue of an element* $\boldsymbol{\varrho} \in \boldsymbol{\zeta}(n)$ *on the Schur module* $Schur_\mu(n)$ *equals the eigenvalue its image* $\mathcal{W}_n\big(\boldsymbol{\varrho}\big)$ *on the Weyl module* $Weyl_\mu(n)$.

5 The limit $n \to \infty$ for $\boldsymbol{\zeta}(n)$: the algebra $\boldsymbol{\zeta}$

5.1 The Capelli monomorphisms $\mathbf{i}_{n+1,n}$

Given $n \in \mathbb{Z}^+$, let we recall that $\boldsymbol{\zeta}(n)$ denote the center of the enveloping algebra $\mathbf{U}(gl(n))$, and

$$\mathbf{H}_k(n), \quad k = 1, \ldots, n$$

denote the Capelli free generators $\boldsymbol{\zeta}(n)$, for every $n \in \mathbb{Z}^+$.

For every $n \in \mathbb{Z}^+$, let

$$\mathbf{i}_{n+1,n} : \boldsymbol{\zeta}(n) \hookrightarrow \zeta(n+1)$$

be the algebra monomorphism:

$$\mathbf{i}_{n+1,n} : \mathbf{H}_k(n) \to \mathbf{H}_k(n+1), \quad k = 1, 2, \ldots, n.$$

We will refer to the monomorphism $\mathbf{i}_{n+1,n}$ as the *Capelli monomorphisms.*

Remark 5.1. *Given $m \in \mathbb{Z}^+$, let $\boldsymbol{\zeta}(n)^{(m)}$ denote the $m-th$ filtration element of $\boldsymbol{\zeta}(n)$ (with respect to the filtration induced by the standard filtration of $\mathbf{U}(n)$).*

Clearly, the Capelli monomorphisms are morphisms in the category of filtered algebras, that is

$$\mathbf{i}_{n+1,n}\Big[\boldsymbol{\zeta}(n)^{(m)}\Big] \subseteq \boldsymbol{\zeta}(n+1)^{(m)}$$

Definition 5.2. *We consider the direct limit (in the category of filtered algebras):*

$$\underrightarrow{lim}\ \boldsymbol{\zeta}(n) = \boldsymbol{\zeta}. \tag{52}$$

The algebra $\boldsymbol{\zeta}$ inherits a structure of filtered algebra, where

$$\boldsymbol{\zeta}^{(m)} = \underrightarrow{lim}\ \boldsymbol{\zeta}(n)^{(m)}.$$

On the other hand, for every $n \in \mathbb{Z}^+$, we may consider the projection operator

$$\boldsymbol{\pi}_{n,n+1} : \boldsymbol{\zeta}(n+1) \twoheadrightarrow \boldsymbol{\zeta}(n),$$

such that

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{H}_k(n+1)) = \mathbf{H}_k(n) \quad k = 1, 2, \ldots, n,$$
$$\boldsymbol{\pi}_{n,n+1}(\mathbf{H}_{n+1}(n+1)) = 0.$$

The following Remarks and Proposition are fairly obvious from the definitions.

Remark 5.3.

1. $$Ker\big(\boldsymbol{\pi}_{n,n+1}\big) = \Big(\mathbf{H}_{n+1}(n+1)\Big),$$

   the bilateral ideal of $\boldsymbol{\zeta}(n+1)$ generated by the element $\mathbf{H}_{n+1}(n+1)$.

2. *The projection $\boldsymbol{\pi}_{n,n+1}$ is the (filtered) left inverse of the Capelli monomorphism $\mathbf{i}_{n+1,n}$.*

   In symbols,

   $$\boldsymbol{\pi}_{n,n+1} \circ \mathbf{i}_{n+1,n} = Id_{\boldsymbol{\zeta}(n)}.$$

Proposition 5.4. *If $n \geq m$, then the restriction $\boldsymbol{\pi}^{(m)}_{n,n+1}$ of $\boldsymbol{\pi}_{n,n+1}$ to $\boldsymbol{\zeta}(n+1)^{(m)}$ and the restriction $\mathbf{i}^{(m)}_{n+1,n}$ of $\mathbf{i}_{n+1,n}$ to $\boldsymbol{\zeta}(n)^{(m)}$ are the inverse of each other.*

Claim 5.5. *The crucial point is that the projections $\boldsymbol{\pi}_{n,n+1}$ admit an intrinsic/invariant presentation that is founded on the Olshanski decomposition.*

5.2 The Olshanski decomposition/projection

We recall a special case of an essential costruction due to Olshanski [57], [59]. For the sake of simplicity, we follow Molev ([49], pp. 928 ff.).

Let $\mathbf{U}(gl(n+1))^0$ be the centralizer in $\mathbf{U}(gl(n+1))$ of the element $e_{n+1,n+1}$ of the standard basis of $gl(n+1)$, regarded as an element of $\mathbf{U}(gl(n+1))$.

Let $\mathbf{I}(n+1)$ be the *left ideal* of $\mathbf{U}(gl(n+1))$ generated by the elements

$$e_{i,n+1}, \quad i = 1, 2, \ldots, n+1.$$

Let $\mathbf{I}(n+1)^0$ be the intersection

$$\mathbf{I}(n+1)^0 = \mathbf{I}(n+1) \cap \mathbf{U}(gl(n+1))^0. \tag{53}$$

We recall that $\mathbf{I}(n+1)^0$ is a *bilateral ideal* of $\mathbf{U}(gl(n+1)^0$, and the following *direct sum decomposition* hold

$$\mathbf{U}((gl(n+1))^0 = \mathbf{U}(gl(n)) \oplus \mathbf{I}(n+1)^0. \tag{54}$$

Therefore, the *Olshanski map*

$$\mathcal{M}_{n+1} : \mathbf{U}((gl(n+1))^0 \twoheadrightarrow \mathbf{U}(gl(n))$$

that maps any element in the direct summand $\mathbf{U}(gl(n))$ to itself and any element in the direct summand $\mathbf{I}(n+1)^0$ to zero is a well-defined algebra epimorphism.

Since $\boldsymbol{\zeta}(n+1)$ is a subalgebra of $\mathbf{U}(n+1)^0$, the direct sum decomposition (54) induces a direct sum decomposition of any element in $\boldsymbol{\zeta}(n+1)$ and the $\mathcal{M}_{n+1}$ map defines, by restriction, an algebra epimorphism

$$\boldsymbol{\mu}_{n,n+1} : \boldsymbol{\zeta}(n+1) \twoheadrightarrow \boldsymbol{\zeta}(n).$$

In plain words, any element $\boldsymbol{\varrho} \in \boldsymbol{\zeta}(n+1)$ admits a *unique* decomposition

$$\boldsymbol{\varrho} = \boldsymbol{\varrho}' \dotplus \boldsymbol{\varrho}^0, \quad \boldsymbol{\varrho}' \in \boldsymbol{\zeta}(n), \;\; \boldsymbol{\varrho}^0 \in \mathbf{I}(n+1)^0. \tag{55}$$

We call the decomposition (55) the *Olshanski decomposition* of the element $\boldsymbol{\varrho} \in \boldsymbol{\zeta}(n)$.

In this notation, the projection

$$\boldsymbol{\mu}_{n,n+1} : \boldsymbol{\zeta}(n+1) \twoheadrightarrow \boldsymbol{\zeta}(n),$$

$$\boldsymbol{\mu}_{n+1,n}(\boldsymbol{\varrho}) = \boldsymbol{\varrho}', \quad \boldsymbol{\varrho} \in \boldsymbol{\zeta}(n+1)$$

is defined.

We claim that

Proposition 5.6.

$$\mathbf{H}_k(n+1) = \mathbf{H}_k(n) \dotplus \mathbf{H}_k(n+1)^0,$$

where

$$\mathbf{H}_k(n+1)^0 = \mathbf{H}_k(n+1) - \mathbf{H}_k(n) \in \mathbf{I}(n+1)^0,$$

and

$$\mathbf{H}_k(n) \in \boldsymbol{\zeta}(n).$$

Example 5.7. We have:

$$\begin{aligned}\mathbf{H}_2(4) &= [21|12] + [31|13] + [41|14] + [32|23] + [42|24] + [43|34] \\ &= \mathbf{H}_2(3) \dotplus \mathbf{H}_2(4)^0,\end{aligned}$$

where

$$\mathbf{H}_2(3) = [21|12] + [31|13] + [32|23] \in \boldsymbol{\zeta}(3),$$

and

$$\mathbf{H}_2(4)^0 = [41|14] + [42|24] + [43|34] \in \mathbf{I}(4)^0.$$

$\square$

Proposition 5.8. *The map* $\boldsymbol{\mu}_{n,n+1}$ *is the same as the map* $\boldsymbol{\pi}_{n,n+1}$.

Therefore, in the following, we refer to the projections

$$\boldsymbol{\mu}_{n,n+1} = \boldsymbol{\pi}_{n,n+1}$$

as the *Capelli-Olshanski projections.*

Remark 5.9. *From Proposition 5.4, the algebra* $\boldsymbol{\zeta}$ *(direct limit) is the same as the projective limit in the category of filtered algebras*

$$\boldsymbol{\zeta} = \varprojlim \boldsymbol{\zeta}(n)$$

with respect to the system of Capelli-Olshanski projections (Molev, [49]).

5.3 Main results

From their virtual presentation, we directly infer the *Olshanski decompositions* and *Capelli-Olshanski projections* :

Proposition 5.10.

1. $\mathbf{I}_k(n+1) = \mathbf{I}_k(n) \dot{+} \mathbf{I}_k(n+1)^0$, *where*

$$\mathbf{I}_k(n+1)^0 = \mathbf{I}_k(n+1) - \mathbf{I}_k(n) \in \mathbf{I}_k(n+1)^0,$$

and

$$\mathbf{I}_k(n) \in \boldsymbol{\zeta}(n).$$

Then

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{I}_k(n+1)) = \mathbf{I}_k(n). \tag{56}$$

2. $\mathbf{K}_\lambda(n+1) = \mathbf{K}_\lambda(n) \dot{+} \mathbf{K}_\lambda(n+1)^0$, *where*

$$\mathbf{K}_\lambda(n+1)^0 = \mathbf{K}_\lambda(n+1) - \mathbf{K}_\lambda(n)\mathbf{I}_k(n+1)^0,$$

and

$$\mathbf{K}_\lambda(n) \in \boldsymbol{\zeta}(n).$$

Then

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{K}_\lambda(n+1)) = \mathbf{K}_\lambda(n). \tag{57}$$

3. $\mathbf{J}_\lambda(n+1) = \mathbf{J}_\lambda(n) \dot{+} \mathbf{J}_\lambda(n+1)^0$, *where*

$$\mathbf{J}_\lambda(n+1)^0 = \mathbf{J}_\lambda(n+1) - \mathbf{J}_\lambda(n) \in \mathbf{I}_k(n+1)^0,$$

and

$$\mathbf{J}_\lambda(n) \in \boldsymbol{\zeta}(n).$$

Then

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{J}_\lambda(n+1)) = \mathbf{J}_\lambda(n). \tag{58}$$

4. $\mathbf{S}_\lambda(n+1) = \mathbf{S}_\lambda(n) \dot{+} \mathbf{S}_\lambda(n+1)^0$, *where*

$$\mathbf{S}_\lambda(n+1)^0 = \mathbf{S}_\lambda(n+1) - \mathbf{S}_\lambda(n) \in \mathbf{I}_k(n+1)^0,$$

and

$$\mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}(n).$$

Then

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{S}_\lambda(n+1)) = \mathbf{S}_\lambda(n). \tag{59}$$

By combining the preceding Proposition with Proposition 5.4, we get

Theorem 5.11. *We have:*

- $\mathbf{I}_k(n) \in \boldsymbol{\zeta}(n)^{(k)}$, *then*

$$\mathbf{i}_{n+1,n}(\mathbf{I}_k(n)) = \mathbf{I}_k(n+1), \quad n \geq k;$$

- $\mathbf{K}_\lambda(n) \in \boldsymbol{\zeta}(n)^{(|\lambda|)}$, *then*

$$\mathbf{i}_{n+1,n}(\mathbf{K}_\lambda(n)) = \mathbf{K}_\lambda(n+1), \quad n \geq |\lambda|;$$

- $\mathbf{J}_\lambda(n) \in \boldsymbol{\zeta}(n)^{(|\lambda|)}$, *then*

$$\mathbf{i}_{n+1,n}(\mathbf{J}_\lambda(n)) = \mathbf{J}_\lambda(n+1), \quad n \geq |\lambda|;$$

- $\mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}(n)^{(|\lambda|)}$, *then*

$$\mathbf{i}_{n+1,n}(\mathbf{S}_\lambda(n)) = \mathbf{S}_\lambda(n+1), \quad n \geq |\lambda|.$$

Passing to the direct limit $\underrightarrow{lim}\ \boldsymbol{\zeta}(n) = \boldsymbol{\zeta}$, we set:

Definition 5.12.

$\mathbf{H}_k = \underrightarrow{lim}\ \mathbf{H}_k(n) \in \boldsymbol{\zeta}, \quad n \geq k.$

$\mathbf{I}_k = \underrightarrow{lim}\ \mathbf{I}_k(n) \in \boldsymbol{\zeta}, \quad n \geq k.$

$\mathbf{K}_\lambda = \underrightarrow{lim}\ \mathbf{K}_\lambda(n) \in \boldsymbol{\zeta}, \quad n \geq |\lambda|.$

$\mathbf{J}_\lambda = \underrightarrow{lim}\ \mathbf{J}_\lambda(n) \in \boldsymbol{\zeta}, \quad n \geq |\lambda|.$

$\mathbf{S}_\lambda = \underrightarrow{lim}\ \mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}, \quad n \geq |\lambda|.$

From the definition of Capelli monomorphisms and Theorem 5.11, it follows

Proposition 5.13. *The elements $\mathbf{H}_k, \mathbf{I}_k, \mathbf{K}_\lambda, \mathbf{J}_\lambda, \mathbf{S}_\lambda \in \boldsymbol{\zeta}$ can be consistently written as formal series. More precisely, setting $L^* = \mathbb{Z}^* = \{1, 2, \ldots\}$,*

- $$\mathbf{H}_k = \sum_{i_1 < \cdots < i_k} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k] = \\ = \sum_{i_1 < \cdots < i_k} \mathfrak{p}\big(e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}\big)$$

- $$\mathbf{I}_k = \sum_{j_1 < j_2 < \cdots < j_p} (i_{j_1}!\ i_{j_2}! \cdots i_{j_p}!)^{-1}\ [j_p^{i_{j_p}} \cdots j_2^{i_{j_2}} j_1^{i_{j_1}} | j_1^{i_{j_1}} j_2^{i_{j_2}} \cdots j_p^{i_{j_p}}]^* = \\ = \sum_{j_1 < j_2 < \cdots < j_p} (i_{j_1}!\ i_{j_2}! \cdots i_{j_p}!)^{-1}\ \mathfrak{p}\big(e_{j_p,\beta}^{i_{j_p}} \cdots e_{j_2,\beta}^{i_{j_2}} e_{j_1,\beta}^{i_{j_1}} e_{\beta,j_1}^{i_{j_1}} e_{\beta,j_2}^{i_{j_2}} \cdots e_{\beta,j_p}^{i_{j_p}}\big),$$

  where $\beta \in A_1$ denotes any negative virtual symbol, the sum is extended to all $p-$tuples $j_1 < j_2 < \cdots < j_p$ in L^* ($p \leq k$), and to all the $p-$tuples of exponents $(i_{j_1}, i_{j_2}, \cdots, i_{j_p})$ such that $i_{j_1} + i_{j_2} + \cdots + i_{j_p} = k$ and any

  $$[j_p^{i_{j_p}} \cdots j_2^{i_{j_2}} j_1^{i_{j_1}} | j_1^{i_{j_1}} j_2^{i_{j_2}} \cdots j_p^{i_{j_p}}]^*$$

  is a permanental Capelli bitableau with one row.

- $$\mathbf{K}_\lambda = \sum_S \mathfrak{p}\big(e_{S,C^*_\lambda} \cdot e_{C^*_\lambda,S}\big) = \sum_S SC^*_\lambda \; C^*_\lambda S,$$

  where the sum is extended to all row-increasing tableaux S on the alphabet L^*.

- $$\mathbf{J}_\lambda = \sum_S (o_S)^{-1} \; \mathfrak{p}\big(e_{S,D^*_{\widetilde{\lambda}}} \cdot e_{D^*_{\widetilde{\lambda}},S}\big) = \sum_S (o_S)^{-1} \; SD^*_{\widetilde{\lambda}} \; D^*_{\widetilde{\lambda}}S,$$

  where the sum is extended to all column-nondecreasing tableaux S (of shape $\widetilde{\lambda}$) on the alphabet L^*.

- $$\begin{aligned} \mathbf{S}_\lambda &= \frac{1}{H(\lambda)} \sum_S \mathfrak{p}\big(e_{S,C^*_\lambda} \cdot e_{C^*_\lambda,D^*_\lambda} \cdot e_{D^*_\lambda,C^*_\lambda} \cdot e_{C^*_\lambda,S}\big) \\ &= \frac{1}{H(\lambda)} \sum_S SC^*_\lambda \; C^*_\lambda D^*_\lambda \; D^*_\lambda C^*_\lambda \; C^*_\lambda S, \end{aligned}$$

  where the sum is extended to all row-increasing tableaux S on the alphabet L^*.

From Proposition 5.9, it follows

Corollary 5.14. *We have:*

$$\varprojlim \mathbf{H}_k(n) = \mathbf{H}_k \in \boldsymbol{\zeta},$$

$$\varprojlim \mathbf{I}_k(n) = \mathbf{I}_k \in \boldsymbol{\zeta},$$

$$\varprojlim \mathbf{K}_\lambda(n) = \mathbf{K}_\lambda \in \boldsymbol{\zeta},$$

$$\varprojlim \mathbf{J}_\lambda(n) = \mathbf{J}_\lambda \in \boldsymbol{\zeta},$$

$$\varprojlim \mathbf{S}_\lambda(n) = \mathbf{S}_\lambda \in \boldsymbol{\zeta}.$$

Due the fact that the algebra $\boldsymbol{\zeta}$ is defined as a direct limit, we infer:

Theorem 5.15.

1. *The set*
$$\left\{\mathbf{H}_k;\ k \in \mathbb{Z}^+\right\}$$
is a system of free algebraic generators of $\boldsymbol{\zeta}$.

2. *The set*
$$\left\{\mathbf{I}_k;\ k \in \mathbb{Z}^+\right\}$$
is a system of free algebraic generators of $\boldsymbol{\zeta}$.

3. *The set*
$$\left\{\mathbf{K}_\lambda;\ \lambda \text{ any partition }\right\}$$
is a linear basis of $\boldsymbol{\zeta}$.

4. *The set*
$$\left\{\mathbf{J}_\lambda;\ \lambda \text{ any partition }\right\}$$
is a linear basis of $\boldsymbol{\zeta}$.

5. *The set*
$$\left\{\mathbf{S}_\lambda;\ \lambda \text{ any partition }\right\}$$
is a linear basis of $\boldsymbol{\zeta}$.

6 The algebra $\Lambda^*(n)$ of shifted symmetric polynomials and the Harish-Chandra Isomorphism

6.1 The Harish-Chandra isomorphism $\chi_n : \boldsymbol{\zeta}(n) \longrightarrow \Lambda^*(n)$

In this subsection we follow Okounkov and Olshanski [56].

As in the classical context of the algebra $\Lambda(n)$ of symmetric polynomials in n variables $x_1, x_2, \ldots, x_n$, the algebra $\Lambda^*(n)$ of *shifted symmetric polynomials* is an algebra of polynomials $p(x_1, x_2, \ldots, x_n)$ but the ordinary symmetry is replaced by the *shifted symmetry*:

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i+1} - 1, x_i + 1, \ldots, x_n),$$

for $i = 1, 2, \ldots, n-1$.

Examples 6.1. Two basic classes of shifted symmetric polynomials are provided by the sequences of *shifted elementary symmetric polynomials* and *shifted complete symmetric polynomials*.

Elementary shifted symmetric polynomials

For every $k \in \mathbb{N}$ let

$$\mathbf{e}^*_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \cdots (x_{i_k}), \quad (60)$$

and $\mathbf{e}^*_0(x_1, x_2, \ldots, x_n) = \mathbf{1}$.

Complete shifted symmetric polynomials

For every $r \in \mathbb{N}$ let

$$\mathbf{h}^*_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 < \cdots \leq i_k \leq n} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots (x_{i_k}), \quad (61)$$

and $\mathbf{h}^*_0(x_1, x_2, \ldots, x_n) = \mathbf{1}$.

Definition 6.2. *The Harish-Chandra isomorphism χ_n is the algebra isomorphism*

$$\chi_n : \boldsymbol{\zeta}(n) \longrightarrow \Lambda^*(n), \qquad A \mapsto \chi_n(A),$$

$\chi_n(A)$ being the shifted symmetric polynomial such that, for every highest weight module V_μ, the evaluation $\chi_n(A)(\mu_1, \mu_2, \ldots, \mu_n)$ equals the eigenvalue of $A \in \boldsymbol{\zeta}(n)$ in V_μ ([56], Proposition **2.1***).*

6.2 The Harish-Chandra images of the Capelli free generators $\mathbf{H}_k(n)$

From Corollary 4.6.1, it follows

Proposition 6.3.

$$\chi_n(\mathbf{H}_k(n)) = \mathbf{e}^*_k(x_1, x_2, \ldots, x_n) \in \Lambda^*(n),$$

for every $k = 1, 2, \ldots, n$.

6.3 The Harish-Chandra images of the Nazarov/Umeda elements $\mathbf{I}_k(n)$

From Theorem 4.25.2, it follows

Proposition 6.4. *For every* $k \in \mathbb{Z}^+$,

$$\chi_n(\mathbf{I}_k(n)) = \mathbf{h}^*_k(x_1, x_2, \ldots, x_n) \in \Lambda^*(n).$$

6.4 The shifted Schur polynomials $\mathbf{s}^*_\lambda(x_1, \ldots, x_n)$ as images of the Schur elements $\mathbf{S}_\lambda(n)$

Recall that, given a variable z and a natural integer p, the symbol $(z)_p$ denotes (see, e.g. [2]) the *falling factorial polynomials*:

$$(z)_p = z(z-1)\cdots(z-p+1), \quad p \geq 1, \qquad (z)_0 = 1.$$

Let λ be a partition, $\lambda_1 \leq n$.
Following [56], set

$$\mathbf{s}^*_\lambda(x_1, \ldots, x_n) = \frac{det\Big[(x_i + n - i)_{\widetilde{\lambda}_i + n - j}\Big]}{det\Big[(x_i + n - i)_{n-j}\Big]},$$

for every $1 \leq i, j \leq n$.

The polynomials $\mathbf{s}^*_\lambda(x_1, \ldots, x_n)$ are shifted symmetric polynomials (in symbols, $\mathbf{s}^*_\lambda(x_1, \ldots, x_n) \in \Lambda^*(n)$) and are called the *shifted Schur polynomials* in n variables.

From the Characterization Theorems for the Schur elements $\mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}(n)$ (see subsection 4.5.3) and the the Characterization Theorems for the shifted Schur polynomials [56], we have:

Theorem 6.5. *For every partition* λ*,* $\lambda_1 \leq n$*,*

$$\chi_n(\mathbf{S}_\lambda(n)) = \mathbf{s}^*_\lambda(x_1, \ldots, x_n).$$

6.5 The fundamental theorems for the algebra $\Lambda^*(n)$

From Theorem 4.5 and Proposition 6.3, it follows

Proposition 6.6. *The set*

$$\Big\{\mathbf{e}^*_k(x_1, x_2, \ldots, x_n);\ k = 1, 2, \ldots, n\Big\}$$

is a set of free algebra generators of the polynomial algebra $\Lambda^*(n)$.

Since, for every $k \in \mathbb{Z}^+$, the *indicator* (top degree homogeneous part) of $\mathbf{h}_k^*(x_1, x_2, \ldots, x_n)$ is the classical complete homogeneous symmetric polynomial $\mathbf{h}_k(x_1, x_2, \ldots, x_n) \in \Lambda(n)$, from the preceding discussions it also follows

Proposition 6.7. *The set*

$$\Big\{\mathbf{h}_k^*(x_1, x_2, \ldots, x_n);\ k = 1, 2, \ldots, n\Big\}$$

is a set of free algebra generators of the polynomial algebra $\Lambda^*(n)$.

Proposition 6.8. *The set*

$$\Big\{\mathbf{s}_\lambda^*(x_1, \ldots, x_n);\ \lambda_1 \leq n\Big\}$$

is a linear basis of the polynomial algebra $\Lambda^*(n)$.

7 The algebra Λ^* of shifted symmetric functions and the Harish-Chandra isomorphism $\chi : \boldsymbol{\zeta} \to \Lambda^*$

7.1 The monomorphism $\mathbf{i}_{n+1,n}^*$ and the projection $\boldsymbol{\pi}_{n,n+1}^*$

Let

$$\mathbf{i}_{n+1,n}^* : \Lambda^*(n) \hookrightarrow \Lambda^*(n+1)$$

be the algebra monomorphism such that

$$\mathbf{i}_{n+1,n}^*\big(\mathbf{e}_k^*(x_1, x_2, \ldots, x_n)\big) = \mathbf{e}_k^*(x_1, x_2, \ldots, x_n, x_{n+1}),$$

for $k = 1, 2, \ldots, n$.

Remark 7.1. *Given $m \in \mathbb{Z}^*$, let $\Lambda^*(n)^{(m)}$ denote the $m-$th filtration element of $\Lambda^*(n)$ (with respect to the filtration induced by the standard filtration of the algebra of polynomials in the variables $x_1, x_2, \ldots, x_n$).*

Clearly, the monomorphisms $\mathbf{i}_{n+1,n}^*$ are morphisms in the category of filtered algebras, that is

$$\mathbf{i}_{n+1,n}^*\Big[\Lambda^*(n)^{(m)}\Big] \subseteq \Lambda^*(n+1)^{(m)}.$$

Definition 7.2. *We consider the direct limit (in the category of filtered algebras):*

$$\varinjlim \Lambda^*(n) = \Lambda^*. \tag{62}$$

The algebra Λ^* inherits a structure of filtered algebra, where

$$\Lambda^{*(m)} = \varinjlim \Lambda^*(n)^{(m)}.$$

Let

$$\boldsymbol{\pi}^*_{n,n+1} : \Lambda^*(n+1) \twoheadrightarrow \Lambda^*(n)$$

be the algebra epimorphism such that

$$\boldsymbol{\pi}^*_{n,n+1}\big(\mathbf{f}^*(x_1, x_2, \ldots, x_n, x_{n+1})\big) = \mathbf{f}^*(x_1, x_2, \ldots, x_n, 0),$$

for every $\mathbf{f}^*(x_1, x_2, \ldots, x_n, x_{n+1}) \in \Lambda^*(n+1)$. Clearly,

$$\boldsymbol{\pi}^*_{n,n+1}\big(\mathbf{e}^*_k(x_1, x_2, \ldots, x_n, x_{n+1})\big) = \mathbf{e}^*_k(x_1, x_2, \ldots, x_n),$$

for $k = 1, 2, \ldots, n$, and

$$\boldsymbol{\pi}^*_{n,n+1}\big(\mathbf{e}^*_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1})\big) = 0.$$

As for the centers $\boldsymbol{\zeta}(n+1)$ and $\boldsymbol{\zeta}(n)$, the following Remarks and Proposition on $\Lambda^*(n+1)$ and $\Lambda^*(n)$ are obvious from the definitions.

Remark 7.3. *We have*

1. $Ker\big(\boldsymbol{\pi}^*_{n,n+1}\big)$ *is the bilateral ideal*

   $$\Big(\mathbf{e}^*_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1})\Big)$$

   of $\Lambda^*(n+1)$ *generated by the element* $\mathbf{e}^*_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1})$.

2. *The projection* $\boldsymbol{\pi}^*_{n,n+1}$ *is the left inverse of the monomorphism* $\mathbf{i}^*_{n+1,n}$. *In symbols,*

   $$\boldsymbol{\pi}^*_{n,n+1} \circ \mathbf{i}^*_{n+1,n} = Id_{\Lambda^*(n)}.$$

Proposition 7.4. *If* $n \geq m$, *then the restriction* ${\boldsymbol{\pi}^*_{n,n+1}}^{(m)}$ *of* $\boldsymbol{\pi}^*_{n,n+1}$ *to* $\Lambda^*(n+1)^{(m)}$ *and the restriction* $\mathbf{i}^*_{n+1,n}$ *of* $\mathbf{i}^*_{n+1,n}$ *to* $\Lambda^*(n)^{(m)}$ *are the inverse of each other.*

Remark 7.5. *From Proposition 7.4, the algebra* Λ^* *(direct limit) is the same as the projective limit in the category of filtered algebras*

$$\Lambda^* = \varprojlim \Lambda^*(n)$$

with respect to the system of the projections $\boldsymbol{\pi}^*_{n,n+1}$, *and therefore, the algebra* Λ^* *is the algebra of shifted symmetric functions of [56].*

7.2 The Harish-Chandra isomorphism $\chi : \boldsymbol{\zeta} \to \Lambda^*$

Consider the following commutative diagram:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boldsymbol{\zeta}^{(m)}(n) \arrow[d, "\chi_n"] \arrow[r, hook, shift right, "\boldsymbol{i}_{n+1,n}"'] & \boldsymbol{\zeta}^{(m)}(n+1) \arrow[d, "\chi_{n+1}"] \arrow[l, two heads, shift right, "\boldsymbol{\pi}_{n,n+1}"'] \\
\Lambda^{*(m)}(n) \arrow[r, hook, shift right, "\boldsymbol{i}^*_{n+1,n}"'] & \Lambda^{*(m)}(n+1) \arrow[l, two heads, shift right, "\boldsymbol{\pi}^*_{n,n+1}"']
\end{tikzcd}
\tag{63}
$$

Theorem 7.6.

If $n \geq m$, the pairs of horizontal arrows in the commutative diagram (63) denote mutually inverse isomorphisms.

Passing to the direct limit, we get the isomorphism of filtered algebras:

$$
\chi : \boldsymbol{\zeta} \approx \Lambda^*.
$$

In particular, we infer the images in Λ^* of the free systems of algebraic generators of $\boldsymbol{\zeta}$:

$$
\Big\{ \mathbf{H}_k;\ k \in \mathbb{Z}^+ \Big\}, \qquad \Big\{ \mathbf{I}_k;\ k \in \mathbb{Z}^+ \Big\}.
$$

with respect to the isomorphism χ.

Proposition 7.7. *We have*

1. *For every $k \in \mathbb{Z}^+$,*

$$
\chi\big(\mathbf{H}_k\big) = \mathbf{e}_k^* \in \Lambda^*,
$$

where

$$
\mathbf{e}_k^* = \sum_{i_1 < i_2 < \cdots < i_k} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \cdots (x_{i_k}), \quad i_s \in \mathbb{Z}^+,
$$

$\mathbf{e}_k^*$ *the $k-th$ elementary shifted symmetric function;*

2. *For every $k \in \mathbb{Z}^+$,*

$$
\chi\big(\mathbf{I}_k\big) = \mathbf{h}_k^* \in \Lambda^*,
$$

where

$$
\mathbf{h}_k^* = \sum_{i_1 \leq i_2 < \cdots \leq i_k} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots (x_{i_k}), \quad i_s \in \mathbb{Z}^+,
$$

$\mathbf{h}_k^*$ *the $k-th$ complete shifted symmetric function.*

7.3 On the isomorphism χ

Notice that, for every $\boldsymbol{\varrho}(n) \in \boldsymbol{\zeta}(n)^{(m)}$, it follows

$$\chi_n(\boldsymbol{\varrho}(n))(x_1, x_2, \ldots, x_n) = {\boldsymbol{\pi}^*_{n,n+1}}^{(m)}\Big(\chi_{n+1}\big({\mathbf{i}^*_{n+1,n}}(\boldsymbol{\varrho}(n))\big)\Big)(x_1, x_2, \ldots, x_n) =$$

$$= \chi_{n+1}\big({\mathbf{i}^*_{n+1,n}}^{(m)}(\boldsymbol{\varrho}(n))\big)(x_1, x_2, \ldots, x_n, 0).$$

Then, for every partition μ and for every $\boldsymbol{\varrho}(n) \in \boldsymbol{\zeta}(n)^{(m)}$, if

$$n \geq max\{m,\ l(\widetilde{\mu}) = \mu_1\},$$

then

$$\chi_n(\boldsymbol{\varrho}(n))(\widetilde{\mu}) = \chi_{n+1}\big({\mathbf{i}^*_{n+1,n}}^{(m)}(\boldsymbol{\varrho}(n))\big)(\widetilde{\mu}).$$

Therefore, the sequence

$$\Big(\ \chi_{n+1}\big({\mathbf{i}^*_{n+1,n}}^{(m)}(\boldsymbol{\varrho}(n))\big)(\widetilde{\mu})\ \Big)_{n \in \mathbb{N}^+}$$

is definitively constant *equal to* $\chi_n(\boldsymbol{\varrho}(n))(\widetilde{\mu})$, and, passing to the direct limit

$$\boldsymbol{\varrho} = \underrightarrow{lim}\ \boldsymbol{\varrho}(n) \in \boldsymbol{\zeta}^{*(m)},$$

Proposition 7.8. *The eigenvalue*

$$\chi(\boldsymbol{\varrho})(\widetilde{\mu}) = \chi_n(\boldsymbol{\varrho}(n))(\widetilde{\mu}), \quad n\ sufficiently\ large \tag{64}$$

is well-defined.

Equation (64) may be regarded as the explicit definition of the isomorphism

$$\chi : \boldsymbol{\zeta} \approx \Lambda^*.$$

7.4 Duality in $\boldsymbol{\zeta}$ and Λ^*

Let

$$\mathcal{W} : \boldsymbol{\zeta} \to \boldsymbol{\zeta}^*$$

denote the automorphism such that

$$\mathcal{W}\Big(\mathbf{H}_k\Big) = \mathbf{I}_k, \quad for\ every\ k \in \mathbb{Z}^+,$$

and let

$$w : \Lambda^* \to \Lambda^*$$

denote the automorphism such that

$$w\Big(\mathbf{e}^*_k\Big) = \mathbf{h}^*_k, \quad for\ every\ k \in \mathbb{Z}^+.$$

Clearly, we have:

Proposition 7.9.

$$\chi \circ \mathcal{W} = w \circ \chi.$$

From Theorem 4.59 and Proposition 7.8, we infer:

Theorem 7.10. *For every $\boldsymbol{\varrho} \in \boldsymbol{\zeta}^{(m)}$ and for every partition μ, we have:*

$$\Big((\chi \circ \mathcal{W})(\boldsymbol{\varrho})\Big)(\mu) = \Big(\chi \circ \boldsymbol{\varrho}\Big)(\widetilde{\mu}). \tag{65}$$

Corollary 7.11.

1. *For every partition λ,*
$$\mathcal{W}\Big(\mathbf{S}_\lambda\Big) = \mathbf{S}_{\widetilde{\lambda}}.$$
2. *In particular,*
$$\mathcal{W}\Big(\mathbf{I}_k\Big) = \mathbf{H}_k, \quad \text{for every } k \in \mathbb{Z}^+;$$
then, the automorphisms $\mathcal{W}$ and w are involutions.

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